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On what has been called Leibniz's rigorous foundation of infinitesimal geometry by means of Riemannian sums

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Abstract

A number of scholars have recently maintained that a theorem in an unpublished treatise by Leibniz written in 1675 establishes a rigorous foundation for the infinitesimal calculus. I argue that this is a misinterpretation.

Zusammenfassung

Eine Reihe von Historikern haben vor kurzem behauptet, dass ein Satz in einer unveröffentlichten Abhandlung von Leibniz, die 1675 geschrieben wurde, eine strenge Grundlage für die Infinitesimalrechnung bildet. Ich behaupte, dass dies eine Fehlinterpretation ist.

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According to what is becoming a standard view among recent Leibniz scholars, an early manuscript by Leibniz, "published in its entirety only very recently," has "radically changed our views on the Leibnizian foundations of the calculus" (Rabouin, 2015, pp. 348–349). According to Knobloch:

> In 1675 . . . Leibniz laid the rigorous foundation of the theory of infinitely small and infinite quantities . . . In modern terms: Leibniz demonstrated the integrability of a huge class of functions by means of Riemannian sums. (Knobloch, 2002, pp. 59, 63)

Arthur quotes this assessment with approval, and elaborates:

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> Leibniz's method, in fact, is extremely general and rigorous; the same construction of elementary and complementary rectangles could be constructed for any curve whatsoever satisfying the three conditions ... continuity, no point of inflection, no point with a vertical tangent. (Arthur, 2008, pp. 24, 21)

Rabouin too agrees enthusiastically:

> We now possess crucial evidence that Leibniz did indeed *demonstrate* ... the equivalence between proofs using infinitesimal methods and proofs using finite quantities ... More than that, the general context of this translation was that of a "rigorous" foundation for the "method of indivisibles" (Leibniz's own terms!). (Rabouin, 2015, p. 364)

Levey is equally convinced:

> The demonstration of Prop. 6 articulates a general technique for finding the quadrature of any continuous curve that contains no point of inflection and no point with a vertical tangent. ... What Leibniz has demonstrated, then, is the integrability of a "huge class of functions." [Levey is quoting Knobloch] ... It goes without saying that his technical accomplishments in quadratures far outstrip the original reaches of the method of exhaustion; the technique of Riemannian integration by itself is an enormous advance, and for Leibniz it is not even particularly a showpiece of [the work in question]. (Levey, 2008, pp. 116, 119)

This interpretation is based on a single theorem: Proposition 6 of a treatise by Leibniz on the arithmetical quadrature of the circle.[1] The above authors all agree that the import of Proposition 6 is that it proves that a general curvilinear area can be approximated with arbitrary precision by rectangles, and that it hence establishes a fully rigorous foundation for integration in general. Let us call this Proposition $6'$.

I shall argue that the $6'$ interpretation is misguided. I say that, first of all, Leibniz's Proposition 6 is about one specific integration formula, not integrability in general, and secondly, that Leibniz didn't think of it as a foundational innovation but as a rather pedantic and basically routine way of applying what is essentially the ancient Greek method of exhaustion.

1. General arguments

My interpretation has considerable prima facie credibility. For if Leibniz had conclusively established the infinitesimal calculus on a fully rigorous foundation already in his twenties, then why did he never publish or refer to this work ever again? He lived for another forty years and had many occasions to write on the foundations of the calculus in print and correspondence, yet he never pointed to this work as establishing the definitive foundations of the calculus.[2] The obvious conclusion would seem to be that this work is not a great foundational masterpiece at all, as is indeed my contention.

The proponents of the $6'$ interpretation address this issue only unconvincingly. Arthur writes off the accumulated evidence of the remaining forty years of Leibniz's life as having "conspired to produce the impression that Leibniz developed his calculus without much attention to its foundations. But this impression is entirely mistaken." (Arthur, 2008, p. 20) He offers no explanation as to how or why so much evidence would have come to conspire to such a supposedly deceptive appearance. Knobloch is similarly unconvincing:

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[1] Leibniz (1993, pp. 28–33), Leibniz (2012, pp. 527–533). The treatise was not published until Knobloch's edition (Leibniz, 1993). It has since been included in the Akademie-Ausgabe of Leibniz's complete works (Leibniz, 2012), and translated into German (Leibniz, 2016) and French (Leibniz, 2004).

[2] Knobloch in Leibniz (1993, pp. 11–14) cites a number of Leibniz's later mentions of this work, none of which have anything to do with Proposition $6'$.