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Duality as a category-theoretic concept

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ABSTRACT

In a paper published in 1939, Ernest Nagel described the role that projective duality had played in the reformulation of mathematical understanding through the turn of the nineteenth century, claiming that the discovery of the principle of duality had freed mathematicians from the belief that their task was to describe intuitive elements. While instances of duality in mathematics have increased enormously through the twentieth century, philosophers since Nagel have paid little attention to the phenomenon. In this paper I will argue that a reassessment is overdue. Something beyond doubt is that category theory has an enormous amount to say on the subject, for example, in terms of arrow reversal, dualising objects and adjunctions. These developments have coincided with changes in our understanding of identity and structure within mathematics. While it transpires that physicists have employed the term 'duality' in ways which do not always coincide with those of mathematicians, analysis of the latter should still prove very useful to philosophers of physics. Consequently, category theory presents itself as an extremely important language for the philosophy of physics.

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1. Introduction

Phenomena covered by the term *duality* have long fascinated mathematicians. While the classification of the five platonic solids is recorded in Book XIII of Euclid's Elements, in what is sometimes called 'Book XV', but believed to be written much later in the 6th century AD by Isidore of Miletus, or perhaps his student, a cube is inscribed in an octahedron and an octahedron inscribed in a cube. This pattern continues, of course, to the other Platonic solids, where the dodecahedron and icosahedron are found to be dual to each other, and the tetrahedron self-dual.

By the middle of the nineteenth century, various 'algebraic' approaches to logic had been developed, and it had been observed that a logical duality obtained on switching propositions and their negations at the same time as switching 'and' and 'or'. For example, De Morgan duality asserts that

- $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$,
- $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$.

Meanwhile in analysis it had been found that problems involving solutions to differential equations could be transformed by Fourier analysis, where the transform of a product of functions is equal to the convolution of the individual transforms, and the transform of the convolution of two functions is the product of the individual transforms.

However, the pinnacle of the nineteenth century interest in duality was reached with projective duality in geometry. Texts would be laid out in parallel columns showing the proofs of dual theorems, with the necessary exchange of 'point' and 'line', 'collinear' and 'concurrent', and so on. For example, we have the following dual theorems, attributed to Pascal and Brianchon:

- Given a hexagon inscribed in a conic section, each of the three pairs of opposite sides determines a point, and these three points are collinear.
- Given a hexagon circumscribed on a conic section, each of the three pairs of opposite vertices determines a line, and these three lines are concurrent.

Duality also came to fascinate physicists through this century. Maxwell understood topics in optics from the perspective of projective geometry, but a more significant manifestation appeared in electromagnetism. Already Faraday had seen that one could anticipate new phenomena by the interchange of electric and magnetic terms. If a fluctuating magnetic field could produce a current in a wire, a fluctuating current should move the needle of a

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nearby compass, indicating a generated magnetic field. This duality was present in Maxwell's own equations for electromagnetism in a vacuum, which reveal invariance under the exchange $E \to B$ and $B \to -E$.

What is notable about these initial appearances of duality is their tendency to broaden, deepen and merge. The duality of the Platonic solids would lead to dual complexes in Poincaré's *analysis situs*, and hence to Poincaré duality, relating aspects of a space in complementary dimensions. Logical duality would lead to inversion of order structures such as lattices, and there merge with similar ideas coming from projective geometry. Pontrjagin dual groups would later be devised to understand Poincaré and other dualities in algebraic topology, and in turn would explain the duality of Fourier analysis. Meanwhile in physics, in his theory of special relativity, Einstein would exploit Maxwellian symmetry which would come to be understood as electric–magnetic duality. In the 1920s Fourier analysis was seen to underlie wave-particle duality of quantum mechanics via the transformation between position space and momentum space. Later in 1931, Dirac seeking a quantum version of electromagnetism was led by electric–magnetic duality to predict the existence of magnetic monopoles.

Right up to the present day, mathematicians' and physicists' fascination with duality shows no sign of abating, from the pure realm of number theory to theoretical physics. For example, we hear that

> It has long been suspected that the Langlands correspondence is somehow related to various dualities observed in quantum field theory and string theory. Both the Langlands correspondence and the dualities in physics have emerged as some sort of non-abelian Fourier transforms. (Geometric Langlands Program Project, 2007)

This is part of an intense interaction between theoretical physicists, mathematical physicists and pure mathematicians, in particular work in the field of 'geometric representation theory'. Analogies between number theory and quantum field theory are widespread, resting on such observations as Michael Atiyah's from the 1970s that the Montonen–Olive dual charge group coincides with the Langlands dual group, and leading to Witten and Kapustin's identification of one side of homological mirror symmetry with one side of the categorical Langlands correspondence, itself understood as a consequence of S-duality (see Frenkel, 2009). Dualities lie at the core of each side of the analogy.

If cutting-edge physics and mathematics have converged on similar structures, what might philosophers of each discipline achieve if they bring their respective backgrounds to think about manifestations of duality? Philosophers of physics have a longstanding interest in situations where two apparently different theories deliver the same empirical predictions. While with gauge equivalent theories it does not seem unreasonable to treat them as variations of the 'same' theory, this appears less plausible in the case of dual string theories. Since mathematics treats dualities between apparently different kinds of mathematical entity, we might expect philosophers of mathematics to be able to be of some service here. However, a search through *The Oxford Handbook of Philosophy of Mathematics and Logic* (Shapiro, 2005) reveals that the phenomenon of duality has made very little impression on the discipline in the Anglophone world. On the other hand, from the perspective of the *philosophy of mathematical practice* (see Mancosu, 2008), if we are to describe the nature of current mathematics, such a central, thematic concept as duality deserves treatment, and, together with Ralf Krömer, I have begun this task (Krömer & Corfield, 2014). That we have an audience in the philosophy of physics should give us great encouragement.

At the very least, from the mathematical side there should be some attempt to convey what kind of thing mathematical duality is, whether it is a circumscribable concept about which it may be possible to forge a general mathematical theory, or rather a much looser, family resemblance kind of notion. A glance at *The Princeton Companion to Mathematics* entry for *duality* may incline us to the latter viewpoint:

> Duality is an important general theme that has manifestations in almost every area of mathematics … Despite the importance of duality in mathematics, there is no single definition that covers all instances of the phenomenon. (Gowers, Barrow-Green, & Leader, 2008, p. 187)

So does mathematical duality shape up to be an exhaustively definable concept, or will it retain an elusive quality, which allows it to manifest itself from time to time in Protean fashion in different portions of mathematics? Well, even if not exhaustible, there is already a theoretical framework in which it is possible to draw together much of what is designated as duality. That framework is provided by category theory, and a major thrust of this paper is to support the idea that the ability to formulate results at such a high level of generality indicates how category theory may provide indispensable insights into the subject matter of mathematics. Set theoretic resources are far too weak in this regard.

Category theory will also provide insight into another notable aspect of the mathematical treatment of duality. While many early forms that we have seen related things of a similar nature – points–lines, functions–functions, groups–groups, logical expressions–logical expressions – later dualities expanded to allow different kinds of entity to be related: theories–models, spaces–quantities. Some have looked to subsume these different faces under the so-called 'Isbell duality' which governs many relationships between geometry and algebra.

As we proceed, we will have to come to understand the differences between physicists' and mathematicians' uses of the word 'duality'. It transpires that these diverge considerably, and yet this should not stand in the way of a dialogue. On the one hand, there is interesting physics to be found employing genuine mathematical duality, while on the other, even if on occasions a case of physical duality is better described as a case of mathematical *equivalence*, we should find that the constructions I describe here are still useful. In particular, there are indications of a close resemblance between string dualities and the so-called 'Morita equivalence' (see Okada, 2009). In his paper, Morita (1958) treated both equivalences ('isomorphisms') of module categories, but also 'dualities' of such categories. They arise through similar constructions where a 'bimodule' mediates between two settings.

I will return to this matter below. First, however, let us set the scene to see what philosophy has had to say about duality in mathematics until now.

2. Philosophers on mathematical duality

To date philosophers have found surprisingly little to say about this feature of mathematics, especially in the Anglophone world. One notable exception was Ernest Nagel who in his 1939 paper explained how the discovery of duality in projective geometry liberated mathematics from the idea that it was dealing with specific elements bearing a set of defining properties. Before we come to look more closely at this paper, it is worth noting that Nagel is dealing here merely with one episode in the history of mathematics' treatment of duality, an episode that had run its course decades earlier. With the further advantage of hindsight three-quarters of a century after Nagel, we should expect new issues to have arisen.