A new characterization of the discrete Sugeno integral

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1. Introduction and preliminaries

Various integral-based operators have become a powerful tool in several applications, mainly in decision theory cf. [3,6–8,11,12,15]. One of the important operators is represented by Sugeno integral, introduced in 1974 by Sugeno [14], which has become a powerful tool in several applications rather soon [6–8].

Among several properties of the Sugeno integrals, the crucial role is played by the fact that they are aggregation functions. Recall that a mapping $A : [0,1]^n \to [0,1]$ is called an aggregation function whenever it is nondecreasing in each coordinate, and satisfies two boundary conditions $A(\emptyset) = A(0, \ldots, 0) = 0$ and $A(\{1, \ldots, 1\}) = 1$. For more details concerning aggregation functions we recommend the monographs [1] and [9].

For $n \in \mathbb{N}$, denote $N = \{1, \ldots, n\}$. The set function $m : 2^N \to [0,1]$ satisfying $m(\emptyset) = 0$, $m(N) = 1$, and being monotone, i.e., $m(E_1) \leq m(E_2)$ whenever $E_1 \subseteq E_2 \subseteq N$, is called a capacity (in some sources, e.g. in [14], $m$ is called a fuzzy measure). The discrete Sugeno integral $Su_m : [0,1]^n \to [0,1]$ with respect to a capacity $m$ is given by

$$Su_m(x) = \vee_{i \in [0,1]} \left( \vee_{i \in [0,1]} (t \wedge m(\{1 \leq i \leq n\}) \wedge \chi_i \geq t) \right).$$

Two equivalent expressions defining the Sugeno integral $Su_m$ are

$$Su_m(x) = \bigvee_{i \in [0,1]} \left( \bigwedge_{i \in [0,1]} m(\{i\}) \wedge \chi_i \right).$$

and

$$Su_m(x) = \bigvee_{i \in [0,1]} \left( x_i \wedge m(\{i\}) \wedge \chi_i \right).$$

where $\cdot : N \to N$ is a permutation such that $x_{1(1)} \leq \cdots \leq x_{n(n)}$. Recall that, for any capacity $m$, the Sugeno integral is

- idempotent, $Su_m(c, \ldots, c) = c$ for any $c \in [0,1]$;
- comonotone maxitive, $Su_m(x \vee y) = Su_m(x) \vee Su_m(y)$ whenever $x$ and $y$ are comonotone, i.e., $(x_i - x_j)(y_i - y_j) \geq 0$ for any $i, j \in N$;
- min-homogeneous, $Su_m(x \wedge (c, \ldots, c)) = Su_m(x) \wedge c$ for any $c \in [0,1]$.

Several other properties of $Su_m$ can be found in [16] or in [5]. Moreover, we have several kinds of axiomatization of the Sugeno integrals [2,4,5,13]. For example, the next claims are equivalent for an aggregation function $A : [0,1]^n \to [0,1]$.

(i) $A$ is comonotone maxitive and min-homogeneous;
(ii) $A$ is a Sugeno integral, i.e., there is a capacity $m$ so that $A = Su_m$ (observe that $m$ is given by $m(E) = A(1_E)$, where $1_E$ is the characteristic function of the set $E$, $1_E(i) = 1$ if $i \in E$ and $1_E(i) = 0$ otherwise);

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(iii) $A$ is median decomposable, i.e., for any $k \in N, x \in [0, 1]^n$ it holds $A(x) = \text{med}(A(x_k), x_k, A(x^k))$, where $(x^k)_{i} = (x^k)_{i} = x_i$ for $i \neq k$ and $(x^k)_{k} = 0, (x^k)_{k} = 1$.

Due to [5] it is known that Sugeno integrals are just weighted lattice polynomials idempotent in 0 and 1. This fact, considering the real unit interval as a lattice $([0, 1], \wedge, \vee)$, has inspired us to look for some other lattice techniques bringing new views on the Sugeno integrals. Namely, we have looked on lattice congruences and compatible aggregation functions, i.e., aggregation functions preserving each lattice congruence. Surprisingly, a new characterization of the Sugeno integrals was obtained. Moreover, the applicability of the Sugeno integral in multicriteria decision support was better clarified.

This short note is organized as follows. In the next section, compatible $n$-ary aggregation functions are studied and their link to Sugeno integrals is shown. In Section 3, the impact of our results to multicriteria decision support is discussed. Finally, some concluding remarks are added.

2. Congruences on $[0, 1]$ and compatible aggregation functions

Recall that roughly, a congruence $C$ on an algebraic structure is an equivalence relation preserving the basic operations in the considered structure. The exact definition in case of lattices is as follows:

**Definition 2.1.** Let $(L, \wedge, \vee)$ be a lattice. A subset $C \subseteq L^2$ is called a congruence on $L$ whenever it is an equivalence relation preserving the join $\lor$ and the meet $\land$, i.e.,

(i) $(a, a) \in C$ for each $a \in L$;
(ii) if $(a, b) \in C$ then also $(b, a) \in C$;
(iii) if $(a, b), (b, c) \in C$ then also $(a, c) \in C$;
(iv) if $(a, b), (c, d) \in C$ then also $(a \lor c, b \lor d) \in C$;
(v) if $(a, b), (c, d) \in C$ then also $(a \land c, b \land d) \in C$.

For more details concerning lattice congruences and the next result we recommend [10]. For the sake of selfcontainedness, we prove the following proposition.

**Proposition 2.2.** The following are equivalent:

(i) $C$ is a congruence on $([0, 1], \wedge, \vee)$;
(ii) there is a partition $\{J_k | k \in K\}$ of $[0, 1]$ such that $J_k$ is an interval for each $k \in K$, and $C = \bigcup_{k \in K} J_k^2$.

Let us note that each singleton is also considered as an interval of $[0, 1]$.

**Proof.** i)⇒ii) For any $x \in [0, 1]$, denote $J_k = \{y \in [0, 1] | (x, y) \in C\}$, i.e., $J_k$ is the $C$-equivalence class containing the element $x$. For any $u, v \in J_k$, $u < v$, and any $z \in [u, v]$, it holds $(u, v), (z, z) \in C$ and thus $(u \lor z, v \lor z) = (z, v) \in C$, and $(u \land z, v \land z) = (u, z) \in C$. Thus $z \in J_k = J_k \cap k$, i.e., $J_k$ is an interval. Now, it is enough to define $k$ as a mid-point of $J_k$, $k = \{k \in [0, 1] \}$ and obviously $C = \bigcup_{k \in K} J_k^2$, where $J_k | k \in K$ is an interval-partition of $[0, 1]$.

ii)⇒i) It is a matter of direct checking only.

Our main interest is to characterize aggregation functions preserving the congruences on $[0, 1]$.

**Definition 2.3.** An aggregation function $A : [0, 1]^n \rightarrow [0, 1]$ is called compatible if it preserves any congruence $C$ of $([0, 1], \wedge, \vee)$, i.e., if for any $x, y \in [0, 1]^n$ such that $(x_1, y_1), \ldots, (x_n, y_n) \in C$ also $(A(x), A(y)) \in C$.

**Theorem 2.4.** Let $A : [0, 1]^n \rightarrow [0, 1]$ be an aggregation function. Then $A$ is compatible if and only if it is a Sugeno integral, i.e., if $A = \text{Sum}$ for some capacity $m$.

**Proof.** Let $A$ be a compatible aggregation function. For $x \in [0, 1]^n$ and $k \in N$, denote $\delta = A(x_k), \alpha = A(x_k)$ and $\beta = A(x^k)$. Due to the monotonocity of $A$, obviously $\delta \leq \beta \leq \beta$. Suppose that $x_k < \delta$ and consider the partition $\{[0, x_k], [x_k, \delta], [\delta, 1]\}$ of $[0, 1]$. Clearly, we have $(x) = (x_k) = x_i$ for $i \neq k$, and $(x_k) = x_k \in [0, x_k]$. $(x_k) = 0 \in [0, x_k]$. Due to the compatibility of $A$ it holds $A(x), A(x_k) \in \delta$. Since $x_k < \delta \leq \beta$ we have

$A(x) = \delta = \text{med}(x_k, \delta, \beta) = \text{med}(x_k, A(x_k), A(x^k))$.

Using a similar reasoning, we obtain in the case $x_k > \delta$ the equality $\beta$. inequalities $\alpha \leq \beta < x_k$ and thus $A(x) = \text{med}(x_k, A(x_k), A(x^k))$ as well. Summarizing all three discussed cases, we have shown the median decomposability of $A$. As $A(0) = 0$ and $A(1) = 1$, due to [4] it follows that $A$ is a Sugeno integral, $A = \text{Sum}$ for some capacity $m$.

Conversely, suppose $A = \text{Sum}_m$ for some capacity $m: 2^N \rightarrow [0, 1]$. Let $C$ be an arbitrary congruence on $([0, 1], \wedge, \vee)$, i.e., an aggregation function $A : L^m \rightarrow L$ is compatible if and only if $A = \text{Sum}_m$ for some $L$-valued capacity $m: 2^N \rightarrow L$.

3. Scale invariant aggregation functions

Consider a bounded chain $L$ as a scale for the score in multicriteria decision linked to $n$ criteria. Then the normed utility function $A : L^n \rightarrow L$ can be seen as an aggregation function. For a bounded chain $L$, the epimorphism $\psi : L \rightarrow L_1$ is a surjective homomorphism, i.e., $L_1 = \{\psi(x) | x \in L\}$, and for any $x, y \in L, x \leq y$, it holds $\psi(x) \leq \psi(y)$. Moreover, $\psi^{-1}(\{a\})$ is an interval partition of $L$. Obviously, if $L : L_1$ is an epimorphism then card$(L) \geq \text{card}(L_1)$. Let us note, that it can be easily verified that in this case $\psi$ is a lattice homomorphism, i.e., it preserves the lattice operations $\wedge$ and $\vee$.

**Example 3.1.** As a particular example consider the real unit interval $L = [0, 1]$.

(i) The decimal half up rounding is related to the scale $L_1 = [0, 0.1 \ldots, 0.9, 1]$. The corresponding epimorphism $\psi : L \rightarrow L_1$ is given by the interval partition $P = \{\psi^{-1}(0), \psi^{-1}(0.1), \ldots, \psi^{-1}(1)\} = \{[0, 0.05], [0.05, 0.15], \ldots, [0.85, 0.95], [0.95, 1]\}$.

(ii) Similarly, the cerimonial half up rounding is related to $L_1 = [0, 0.01, \ldots, 0.99, 1]$, and $P = \{[0, 0.005], [0.005, 0.015], \ldots, [0.985, 0.995], [0.995, 1]\}$.

(iii) For the linguistic scale $L_1 = \{\text{bad}, \text{medium}, \text{good}, \text{excellent}\}$ one can consider, e.g., $P = \{[0, 0.3], [0.3, 0.7], [0.7, 0.9], [0.9, 1]\}$.

Having an aggregation function $A$ on $L$, its desirable property is the compatibility with the above mentioned epimorphisms.

**Definition 3.2.** Let $(L, \leq)$ be a bounded chain, and let $A : L^n \rightarrow L$ be an aggregation function on $L$. Then $A$ is called scale invariant whenever for any bounded chain $(L_1, \leq_{L_1})$ and epimorphism $\psi : L \rightarrow L_1$ there...
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