Information Fusion 29 (2016) 98–104

Contents lists available at ScienceDirect

Information Fusion

journal homepage: www.elsevier.com/locate/inffus

# Admissible orders of typical hesitant fuzzy elements and their application in ordered information fusion in multi-criteria decision making

Hai Wang, Zeshui Xu*

*School of Economics and Management, Southeast University, Nanjing, Jiangsu 211189, China*

ARTICLE INFO

*Article history:*
Available online 15 September 2015

*Keyword:*
Admissible order
Hesitant fuzzy elements
Hesitant fuzzy sets
OWA operator
Partial orders

ABSTRACT

Typical hesitant fuzzy elements (HFEs) are quite useful for multi-criteria decision making (MCDM) in hesitant fuzzy setting. To reach a decision, it is necessary to derive the orders of HFEs. However, all the existing orders presented for HFEs in the literature are partial orders. We may need total orders sometimes such as in the situations when aggregating information by the ordered weighted aggregation (OWA) operators. Thus, the first purpose of this paper is to develop the total orders (called admissible orders) of HFEs for MCDM. The admissible orders improve the existing partial orders of HFEs and can be generated by a set of special functions. We demonstrate that the distinct ranking of HFEs can be derived according to different admissible orders. Another purpose is to redefine the hesitant fuzzy OWA operator based on the proposed total orders. Some interesting properties of the operator are also discussed.



## 1. Introduction

The concept of fuzzy sets (FSs) [1] has been widely accepted and applied in many categories of decision making problems because of the nature that everything is allowed to be a matter of degrees in fuzzy logic [2]. The prominent characterization of a FS is the considerations of membership degrees to the elements. In real-life situations, the decision making problems contain various kinds of uncertainties [3,4]. For the purpose of handling these uncertainties by the idea of FSs, the extensions of FSs as well as their applications have been an increasing interest in recent years. In quantitative situations, some popular extensions are type-2 fuzzy sets [5], type-n fuzzy sets [6], interval-valued fuzzy sets [2], Atanassov's intuitionistic fuzzy sets [7], fuzzy multisets [8], hesitant fuzzy sets (HFSs) [9] and Pythagorean Fuzzy Sets [10]. The applications of these extensions can be seen in many fields such as image processing [11,12], classification [13,14], text mining [15] and multi-criteria decision making (MCDM) [16–26] and so on.

MCDM refers to evaluating, prioritizing or selecting over some available alternatives $\{A_1, A_2, \ldots, A_p\}$ with respect to a set of criteria $\{c_1, c_2, \ldots, c_q\}$ which are usually conflicted with each other. In order to do that, it is necessary to assign a value to each alternative with respect to each criterion. Then two phases are utilized to construct a ranking among alternatives: the aggregation phase and the exploitation phase [27]. The first phase is usually conducted by employing the aggregation operators with some specific characteristics, such as the ordered weighted aggregation (OWA) operators, to obtain a collective preference structure. Then, a method is considered to derive a ranking of alternatives in the second phase. We argue that a total order is often needed in both phases, if the OWA operators are used in the first phase, or if the alternatives are ranked by the collective performances obtained in the first phase directly.

Among the extensions of FSs, HFSs have attracted the attention of a great deal of researchers in recent few years. A lot of literature has shown that the hesitant situations are very common in practical problems and the HFSs can facilitate the management of hesitant uncertainties. Some authors [28–30] focused on the aggregation of HFSs and developed a series of aggregation operators, such as the hesitant fuzzy ordered weighted aggregation (HFOWA) operator, for MCDM. Hesitant fuzzy preference relations have also been investigated in refs. [31,32]. Some other studies [33,34] presented various decision making approaches in this setting. Besides, several extensions of HFSs have also been presented, such as interval-valued HFSs [26], generalized HFSs [4], dual HFSs [35] and hesitant fuzzy linguistic term sets [36].

However, as a basic theory of using HFSs in decision making, the total orders of hesitant fuzzy elements (HFEs) are not emphasized enough. Xu and Xia [29] defined a score function of a HFE and used it to compare any two HFEs. Another score function can be found in

* Corresponding author. Tel.: +86 2584483382.
*E-mail addresses:* wanghai17@sina.com (H. Wang), xuzeshui@263.net (Z. Xu).

http://dx.doi.org/10.1016/j.inffus.2015.08.009
1566-2535/

ref. [34]. To overcome the limitation of the comparison law only based on the score function, Chen et al. [37] and Liao et al. [38] defined, respectively, distinct forms of deviation degrees. Then HFEs can be ordered by their score functions and deviation degrees. As will be illustrated in Section 3, all these orders are partial orders on a collection of HFEs. But sometimes we may need a total order of HFEs in MCDM, such as in the situations when aggregating information by the HFOWA operator.

Therefore, the first purpose of this paper is to develop the total orders of HFEs. The total order is presented as an admissible order which is a linear order and refines the common partial order of the vectors of dimension $n$. The proposed admissible orders can be generated by a set of special functions. This construction method allows us to build different total orders that improve the existing partial orders between HFEs.

The other purpose of this paper is to use the presented linear orders to the famous OWA operators. We define a general form of the HFOWA operator based on the proposed admissible orders. Some of its properties are also investigated. Based on several novel linear operations of HFEs, we demonstrate the increasing monotonicity of the HFOWA operator.

The rest of the paper is organized as follows: we review some preliminaries of HFSs and poset in Section 2. We present the admissible orders of HFEs in Section 3, as well as the construction method and the properties of the admissible orders. We discuss the HFOWA operator in Section 4. The paper ends with the conclusions.

## 2. Preliminaries

### 2.1. Hesitant fuzzy sets and hesitant fuzzy elements

To handle the case where the experts have hesitancy on several possible values for defining the membership of an object, Torra [9] proposed the concept of hesitant fuzzy set (HFS) whose membership function is described by a subset of [0, 1]. For the purpose of actual applications, Torra [9] also extended the HFS to permit us to have repeated memberships for a given object.

**Definition 1** [9]**.** Let $X$ be a reference set, then we define (multiset based) hesitant fuzzy set (HFS) on $X$ in terms of a function $h$ that when applied to $X$ returns a multisubset of [0, 1].

Recently, Bedregal et al. [30] defined a particular case of HFSs, i.e., the typical hesitant fuzzy sets, which consider only the finite nonempty subsets of [0, 1]. In fact, most of the current studies focus on typical HFSs. In this paper, we consider only the finite and nonempty HFSs although the original names, HFSs and HFEs are used.

**Remark 1.** The idea of (typical) HFSs is close to the fuzzy multisets with a fixed number of memberships mentioned in ref. [39] and the $n$-dimensional fuzzy sets proposed in ref. [40].

Xia and Xu [29] presented the following mathematical representation of a HFS:

$$E = \{< x, h_E(x) > |x \in X\}, \tag{1}$$

where the so-called hesitant fuzzy element (HFE) $h_E(x)$ is a finite nonempty subset (or multisubset) in [0, 1], representing the possible membership degrees of $x$ to the set $E$.

Without loss of generality, the members in a HFE are arranged in an increasing order. In this paper, a HFE is denoted by $h = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$, where $0 \le \gamma_1 \le \gamma_2 \le \cdots \le \gamma_n \le 1$, $n$ is the cardinality of $h$.

Given $x \in X$, the cardinalities of two HFEs $h_1(x)$ and $h_2(x)$ may not coincide. For the convenience of computation, some methods have been considered to make their cardinalities equal [32]. A common way is the use of $\beta$-normalization. If $\#h_1(x) < \#h_2(x)$, then new elements $\bar{h}$ derived by

$$\bar{h} = \zeta h^+ + (1-\zeta)h^- \tag{2}$$

can be appended to $h_1(x)$, where $h^+$ and $h^-$ are respectively the maximum and minimum elements in $h_1(x)$. The parameter $\zeta$ can be seen as an index of risk as in refs. [33,41] and determined by the optimized algorithms as in ref. [32]. As $h_1(x)$ is a multisubset of [0, 1], it is meaningful as well even if Eq. (2) does not generate new elements.

In group decision making circumstances, hesitancies usually emerge because individuals have distinct and uncompromising opinions and the decision makers do not prefer to consider an averaging value by the aggregation operators. In this case, if individuals are asked to provide one value with the highest confident level, the cardinalities of HFEs can be retained the same. We will conduct the following discussions with the assumption that the cardinalities of two HFEs are equal.

### 2.2. Partial orders and posets

We recall some necessary concepts in refs. [42,43] in this section.

For some $n > 0$, the set of all HFEs with the cardinality $n$ is denoted by

$$H([0,1], n) = \{h = \{\gamma_1, \gamma_2, \ldots, \gamma_n\} | 0 \le \gamma_1 \le \gamma_2 \le \cdots \le \gamma_n \le 1\}, \tag{3}$$

and a special kind of set of the vectors with the dimension $n$ is denoted by

$$K([0,1], n) = \{(x_1, x_2, \ldots, x_n) | 0 \le x_1 \le x_2 \le \cdots \le x_n \le 1\}. \tag{4}$$

Observe that there is a bijection $i$ from $H([0, 1], n)$ to $K([0, 1], n)$, given by $i(\{\gamma_1, \gamma_2, \ldots, \gamma_n\}) = (\gamma_1, \gamma_2, \ldots, \gamma_n)$.

Given a nonempty set $H$, a partial order (or weak partial order) $\preccurlyeq$ on the set $H$ is a binary relation on $H$ satisfying:

(1) **Reflexivity:** for any $a \in H$, $a \preccurlyeq a$;
(2) **Antisymmetry:** for all $a, b \in H$, if $a \preccurlyeq b$ and $b \preccurlyeq a$, then $a = b$;
(3) **Transitivity:** for all $a, b, c \in H$, if $a \preccurlyeq b$ and $b \preccurlyeq c$, then $a \preccurlyeq c$.

A set $H$ with a partial order $\preccurlyeq$ is called a partially ordered set (poset) and denoted by $(H, \preccurlyeq)$. Furthermore, for all $a, b, c \in H$, a strict partial order $\prec$ is a binary relation that satisfies:

(1) **Irreflexivity:** not $a \prec a$,
(2) **Transitivity:** if $a \prec b$ and $b \prec c$, then $a \prec c$,
(3) **Asymmetry:** if $a \prec b$ then not $b \prec a$.

There is a one-to-one mapping between each pair of weak and strict partial orders. In fact, if $\preccurlyeq$ is a weak partial order, then the corresponding strict partial order $\prec$ is the irreflexive kernel given by: $a \prec b$ if $a \preccurlyeq b$ and $a \neq b$; conversely, if $\prec$ is a strict partial order, then the corresponding weak partial order $\preccurlyeq$ is the reflexive closure given by: $a \preccurlyeq b$ if $a \prec b$ or $a = b$.

If any two elements $a, b \in H$ such that $a \preccurlyeq b$ or $b \preccurlyeq a$, then the partial order $\preccurlyeq$ is called a linear order and $H$ is called a chain. If the element $a \in H$ such that $x \preccurlyeq a$ holds for any $x \in H$, then $a$ is the top of a poset $(H, \preccurlyeq)$ and denoted by $1_H$. Similarly, the bottom of a poset $(H, \preccurlyeq)$, denoted by $0_H$, represents the smallest element of $H$ with respect to $\preccurlyeq$.

Given a poset $(H, \preccurlyeq)$ with the top $1_H$ and the bottom $0_H$, then $\preccurlyeq$-aggregation function [44] is an aggregation function $f : H^n \to H$ on $H$ with respect to the order $\preccurlyeq$ satisfying:

(1) $f(0_H, 0_H, \ldots, 0_H) = 0_H$, $f(1_H, 1_H, \ldots, 1_H) = 1_H$;
(2) $f(x_1, x_2, \ldots, x_n) \preccurlyeq f(y_1, y_2, \ldots, y_n)$ whenever $(x_1, x_2, \ldots, x_n) \preccurlyeq (y_1, y_2, \ldots, y_n)$,