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Super- and subadditive constructions of aggregation functions

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ABSTRACT

Two construction methods for aggregation functions based on a restricted a priori known decomposition set and decomposition weighing function are introduced and studied. The outgoing aggregation functions are either superadditive or subadditive. Several examples, including illustrative figures, show the potential of the introduced construction methods. Our approach generalizes several known constructions and optimization methods, including decomposition and superdecomposition integrals. We present also an economic applications of the introduced concepts.

1. Introduction

Aggregation functions play an important role in many domains where an n-dimensional input representation is represented by a single value. For more information and details we recommend monographs [1,5] . Recall that for $n \in N$ a monotone function A: $[0,\ 1]^n \to [0,\ 1]$ is called an aggregation function whenever it satisfies two boundary conditions $A(0, \ldots, 0) = A(\mathbf{0}) = 0$ and $A(1, \ldots, 1) = A(\mathbf{1}) = 1$. Observe that we will not consider the usual convention $A(x) = x$ for 1-dimensional aggregation functions. Note also that, in general, some other interval I can be considered instead of the unit interval [0, 1]. However, our results related to [0, 1] domain can be easily generalized to the domain I.

In several practical situations, the aggregation function A is not known on its full domain $[0,\ 1]^n$, but only on a subdomain $\mathcal{H} \subseteq [0, 1]^n$. More often the boundary condition $A(\mathbf{1}) = 1$ is not important, i.e., A and λA gives the same information for the user, independently of $\lambda \in\ ]0,\ \infty[$. This is, e.g., the case when A is considered as a utility function. The above facts have inspired us to introduce two construction methods for aggregation functions when only a partial information is known. Our approach was motivated by the ideas from [6,7] dealing with superadditive and subadditive transformations of aggregation functions on $[0,\ \infty[$. Recall that a function F: $[0,\ \infty[^n \to [0,\ \infty[$ is called superadditive (subadditive) whenever, for any $\mathbf{x}, \mathbf{y} \in [0,\ \infty[^n$, it holds $F(\mathbf{x}+\mathbf{y}) \geq F(\mathbf{x}) + F(\mathbf{y})$ $\ (F(\mathbf{x}+\mathbf{y}) \leq F(\mathbf{x}) + F(\mathbf{y}))$. F is additive if and only if it is both superadditive and subadditive, i.e., $F(\mathbf{x}+\mathbf{y}) = F(\mathbf{x}) + F(\mathbf{y})$. If F is defined on some subdomain $I^n \in [0,\ \infty[^n$; then the above inequalities (equalities) are considered for $\mathbf{x}, \mathbf{y} \in I^n$ such that also $\mathbf{x}+\mathbf{y} \in I^n$.

Our contribution is organized as follows. In Section 2, based on a decomposition set $\mathcal{H}$ and weighing function B, we introduce superadditive and subadditive functions B^* and B_*, and the related aggregation functions $A^{\mathcal{H},B}$ and $A_{\mathcal{H},B}$, including two motivating examples and some preliminary results. In Section 3, we exemplify the functions B^* and B_* for several decomposition pairs $(\mathcal{H}, B)$ and show the link of our constructions to decomposition and superdecomposition integrals [9,10]. In Section 4 we present an economic application showing how the introduced concepts permits to define and measure the utilization rate of the production capacity of a firm. Finally, some concluding remarks are added.

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2. Super- and subadditive constructions of aggregation functions

Fix $n \in N = \{1, 2, \ldots\}$. In what follows, an arbitrary subset $\mathcal{H}$ of $[0, 1]^n$ such that $\mathbf{0} \in \mathcal{H}$ will be called a *decomposition set*. A function $B : \mathcal{H} \to [0, 1]$, not identically equal to zero, with $B(\mathbf{0}) = 0$ and such that $B(\mathbf{x}) \le B(\mathbf{y})$ whenever $\mathbf{x} \le \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \mathcal{H}$, will be called a *decomposition weighing function*. For any subset $S \subseteq [0, \infty[$ of non-negative real values, we will denote by $\inf S$ the greatest lower bound of S, and by $\sup S$ the smallest upper bound. If S is unbounded then $\sup S = \infty$ by convention. Moreover, the convention that $\inf \emptyset = \infty$ and $\sup \emptyset = 0$ will be also considered.

Although a decomposition weighing function is defined only on $\mathcal{H}$ which, in the extreme case, may consist besides $\mathbf{0}$ just of a single point, one may introduce its transformation to the entire unit n-cube $[0, 1]^n$ by letting

$$B_*(\mathbf{x}) = \inf\left\{\sum_{i=1}^{k} B(\mathbf{y}^{(i)}) \mid k \in N,\ (\mathbf{y}^{(i)})_{i=1}^{k} \in \mathcal{H}^k;\ \sum_{i=1}^{k} \mathbf{y}^{(i)} \ge \mathbf{x}\right\} \quad (1)$$

and $$B^*(\mathbf{x}) = \sup\left\{\sum_{i=1}^{k} B(\mathbf{y}^{(i)}) \mid k \in N, (\mathbf{y}^{(i)})_{i=1}^{k} \in \mathcal{H}^k; \sum_{i=1}^{k} \mathbf{y}^{(i)} \le \mathbf{x}\right\}. \quad (2)$$

Observe that, in general, B_* and B^* are mappings from $[0, 1]^n \to [0, \infty]$. The pair $(\mathcal{H}, B)$ will be called *subadmissible* if $B_*(1) \in]0, \infty[$, and *superadmissible* if $B^*(1) \in]0, \infty[$. The set of all subadmissible and superadmissible pairs will be denoted simply by Sub_n and $Super_n$, respectively.

For any subadmissible (superadmissible) pair $(\mathcal{H}, B)$ we may introduce normalized versions of the transformation of B introduced above by letting

$$A_{\mathcal{H},B} : \ [0, 1]^n \to [0, 1];\ \ \mathbf{x} \mapsto B_*(\mathbf{x})/B_*(\mathbf{1}) \quad (3)$$

and

$$A^{\mathcal{H},B} : \ [0, 1]^n \to [0, 1];\ \ \mathbf{x} \mapsto B^*(\mathbf{x})/B^*(\mathbf{1}), \quad (4)$$

where in both cases $\mathbf{1} \in [0, 1]^n$ denotes the all-one vector.

Let us give two economic examples of possible applications of normalized subadmissible and superadmissible normalized transformations $A_{\mathcal{H},B}$ and $A^{\mathcal{H},B}$.

Example 1. Let us suppose that function B is a production function (see e.g., [4,11]) related to a given product so that from the vector of input quantities $\mathbf{x} = [x_1, \ldots, x_n] \in \Re_+^n$ the quantity $B(\mathbf{x}) \in \Re$ is obtained. More precisely, one can imagine that there is a set of admissible input vectors $\mathcal{H} \subseteq \Re_+^n$, so that, in fact, one can imagine the production function as mapping from $\mathcal{H}$ to $\Re_+$. One can also suppose that the input quantities are normalized so that $\mathbf{x} \in [0, 1]^n$ and, consequently, $\mathcal{H} \subseteq [0, 1]^n$. Also the output can be normalized in the interval [0, 1]. Considering that it could be possible to get a greater output by splitting the production related to a vector of input $\mathbf{x} = [x_1, \ldots, x_n] \in [0, 1]$ in the family of vector of inputs $\mathbf{y}^{(i)} \in \mathcal{H}^k, i = 1, \ldots, k$ with $\sum_{i=1}^{k} \mathbf{y}^{(i)} \le \mathbf{x}$ obtaining as output $\sum_{i=1}^{k} B(\mathbf{y}^{(i)})$, by means of the superadditive transformation we get that the maximal output is given by $B^*(\mathbf{1})$. Therefore, the normalized production function related to basic production function B and to the set of admissible input vectors $\mathcal{H}$ is given by $A^{\mathcal{H},B} = B^*(\mathbf{x})/B^*(\mathbf{1})$.

Example 2. Let us consider a financial market (see e.g., [3]) where uncertainty is represented by a set of states $S = \{s_1, \ldots, s_n\}$. States from S are exhaustive and mutually exclusive so that only one state will be true. In this context each vector $\mathbf{x} = [x_1, \ldots, x_n] \in \Re_+^n$ can be considered as a feasible security that pays an outcome $x_i, i = 1, \ldots, n$, if the state s_i is revealed true. Suppose that on the market a set of securities $\mathcal{H} \subset \Re_+^n$ is available. In this context $B : \mathcal{H} \to \Re_+$ is a price function. Fix a vector of outcomes $\mathbf{x} = [x_1, \ldots, x_n] \in \Re_+^n$. A super-replication portfolio ([2]) is a set of securities $\mathbf{y}^{(i)} \in \mathcal{H}, i = 1, \ldots, k,$ such that $\sum_{i=1}^{k} \mathbf{y}^{(i)} \ge \mathbf{x}$. Among all the super-replication portfolios, one economic operators look for that one with the minimum price which is given by $B_*(\mathbf{x})$. One can suppose that all outcomes of considered securities can be normalized so that they take value in [0, 1], and one has B: $[0, 1]^n \to [0, 1]$. Also the prices can be normalized in the interval [0, 1]. In fact, in this context the maximal attainable vector of outcomes is $\mathbf{1}$ having $B^*(\mathbf{1})$ as minimal price of the super replication portfolio. Therefore the prices of portfolio $\mathbf{x} \in [0, 1]^n$ in the considered financial market is given by $A_{\mathcal{H},B} = B_*(\mathbf{x})/B_*(\mathbf{1})$.

Quite expectedly, the introduced functions B_* and B^* as well as their normalized versions $A_{\mathcal{H},B}$ and $A^{\mathcal{H},B}$, are subadditive and superadditive, respectively:

Proposition 1. *If $(\mathcal{H}, B)$ is a subadmissible pair, then $A_{\mathcal{H},B}$ is a subadditive aggregation function. Analogously, if $(\mathcal{H}, B)$ is a superadmissible pair, then $A^{\mathcal{H},B}$ is a superadditive aggregation function.*

Proof. Because of subadmissibility and superadmissibility assumptions, the functions $A_{\mathcal{H},B}$ and $A^{\mathcal{H},B}$ are well defined. Monotonicity of both $A_{\mathcal{H},B}$ and $A^{\mathcal{H},B}$ follow from the monotonicity and non-negativity of decomposition weighing functions. Clearly, $A_{\mathcal{H},B}(\mathbf{0}) = \mathbf{0}$ ($A^{\mathcal{H},B}(\mathbf{0}) = \mathbf{0}$) and $A_{\mathcal{H},B}(\mathbf{1}) = \mathbf{1}$ ($A^{\mathcal{H},B}(\mathbf{1}) = \mathbf{1}$). It remains to prove sub- and superadditivity, and it is clearly sufficient to do this for B_* and B^*. The proof that B_* and B^* are subadditive and superadditive is given in Propositions 3 and 2 , respectively, of [7] .

For arbitrary $\mathbf{x}, \mathbf{y} \in [0, 1]^n$ let $(\bar{\mathbf{x}}^{(i)})_{i=1}^{k}$ and $(\bar{\mathbf{y}}^{(j)})_{j=1}^{\ell}$ be a k-tuple and an ℓ-tuple of vectors in $\mathcal{H}$ for which $\sum_{i=1}^{k} \bar{\mathbf{x}}^{(i)} \ge \mathbf{x}$ and $\sum_{j=1}^{\ell} \bar{\mathbf{y}}^{(j)} \ge \mathbf{y}$. Since, by the choice of these k- and ℓ-tuples, the sum of the vectors in the $(k+\ell)$-tuple $(\bar{\mathbf{x}}^{(1)}, \ldots, \bar{\mathbf{x}}^{(k)}, \bar{\mathbf{y}}^{(1)}, \ldots, \bar{\mathbf{y}}^{(\ell)})$ is at least $\mathbf{x}+\mathbf{y}$, it follows by the definition of B_* that

$$B_*(\mathbf{x}+\mathbf{y}) \le \sum_{i=1}^{k} B(\bar{\mathbf{x}}^{(i)}) + \sum_{j=1}^{\ell} B(\bar{\mathbf{y}}^{(j)}) \ .$$

Now, it is evident that $B_*(x+y) \le B_*(\mathbf{x}) + B_*(\mathbf{y})$.

Similarly, for any $\mathbf{x}, \mathbf{y} \in [0, 1]^n$ let $(\bar{\mathbf{x}}^{(i)})_{i=1}^{k}$ and $(\bar{\mathbf{y}}^{(j)})_{j=1}^{\ell}$ be a k-tuple and an ℓ-tuple of vectors in $\mathcal{H}$ for which $\sum_{i=1}^{k} \bar{\mathbf{x}}^{(i)} \le \mathbf{x}$ and $\sum_{j=1}^{\ell} \bar{\mathbf{y}}^{(j)} \le \mathbf{y}$. By the choice of these k- and ℓ-tuples, the sum of the vectors in the $(k+\ell)$-tuple $(\bar{\mathbf{x}}^{(1)}, \ldots, \bar{\mathbf{x}}^{(k)}, \bar{\mathbf{y}}^{(1)}, \ldots, \bar{\mathbf{y}}^{(\ell)})$ is this time at most $\mathbf{x}+\mathbf{y}$, and so from the definition of B^* we have

$$B^*(\mathbf{x}+\mathbf{y}) \ge \sum_{i=1}^{k} B(\bar{\mathbf{x}}^{(i)}) + \sum_{j=1}^{\ell} B(\bar{\mathbf{y}}^{(j)}) \ .$$

Again, it is evident that $B^*(x+y) \ge B^*(\mathbf{x}) + B^*(\mathbf{y})$. This implies sub- and superadditivity of B_* and B^* and completes the proof. □

We illustrate our proposals in the next simple example. Let $n = 1$ and consider a trivial decomposition system $\mathcal{H} = \{0, 1/t\}$ for some fixed positive integer t. Further, let B be a decomposition weighing function defined by $B(0) = 0$ and $B(1/t) = b > 0$. Obviously, $B_*(0) = 0$. For any $x \in]0, 1]$, letting $k = \lceil tx \rceil$ (the ceiling of tx) we have $x \in](k-1)/t, k/t]$, so that $B_*(x) = kb$ and hence $B_*(1) = tb$; it follows that $A_{\mathcal{H},B}(x) = B_*(x)/B_*(1) = \lceil tx \rceil/t$, which is a subadditive aggregation function. By the same token, letting $\ell = \lfloor tx \rfloor$ (the floor of tx) we have $x \in [\ell/t, (\ell+1)/t[$, so that $B^*(x) = \ell b$, $B^*(1) = tb$, and $A^{\mathcal{H},B}(x) = B^*(x)/B^*(1) = \lfloor tx \rfloor/t$, which is a superadditive aggregation function.

Proposition 2. *If $(\mathcal{H}, B)$ is a subadmissible pair, then $A_{\mathcal{H},B} = B$ if and only if $\mathcal{H} = [0, 1]^n$ and B is subadditive, with $B(\mathbf{1}) = 1$. Analogously, if $(\mathcal{H}, B)$ is a superadmissible pair, then $A^{\mathcal{H},B} = B$ if and only if $\mathcal{H} = [0, 1]^n$ and B is superadditive, with $B(\mathbf{1}) = 1$.*