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Uncertainty characterization of the orthogonal Procrustes problem with arbitrary covariance matrices

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ABSTRACT

This paper addresses the weighted orthogonal Procrustes problem of matching stochastically perturbed point clouds, formulated as an optimization problem with a closed-form solution. A novel uncertainty characterization of the solution of this problem is proposed resorting to perturbation theory concepts, which admits arbitrary transformations between point clouds and individual covariance and cross-covariance matrices for the points of each cloud. The method is thoroughly validated through extensive Monte Carlo simulations, and particularly interesting cases where nonlinearities may arise are further analyzed.

1. Introduction

The problem of finding the similarity transformation between two sets of points in n-dimensional space appears commonly in many applications of computer vision, robotics, statistics, and other fields of research. The study of this family of problems is usually known as the Procrustes analysis [1], which includes the statistical characterization of the transformation between the shape of objects [2]. One particularly important problem in this family is the so-called orthogonal Procrustes problem, which can be traced back to the work presented in [3], and consists in extracting the orthogonal transformation that maps one set of points into a second set of points, with known associations between them. It is closely related to Wahba's problem [4] and to the Kabsch algorithm [5]. The generalization for rotation, translation, and scaling has been the subject of extensive research in areas such as computer vision, and can be traced back to [6–8]. While initially the problem was posed without any restrictions on the transformation between the sets, i.e., rotations and reflections were allowed, a more evolved strategy appeared restricting the transformation to the special orthogonal group, as detailed in [6,9]. Furthermore, Goryn and Hein [10] demonstrated that the previous solutions are optimal even when both data sets are perturbed with isotropic and identical Gaussian noise.

The statistical characterization of the Procrustes analysis has also been the subject of study in works such as [2,9,11,12]. Using perturbation theory, the nonlinear problem of characterizing the uncertainty was addressed with some limiting options, such as the absence of weighting of the point sets, the use of small rotations, or the same covariance for all points. Within the field of medical imaging, the work presented in [13] also resorts to perturbation theory to present a statistical characterization of a target position, considering small rotations, isotropic uncertainty, and equal weights for each point. More recently, the work presented in [14] extends these results for anisotropic uncertainty in the components of the point space. This is achieved by considering the same covariance matrix for all points, which may weigh each component of the point space independently. The authors of [15] further expand this by considering different noise levels for each point, while keeping the linearized model for the rotation matrix. An interesting advance in the study of the uncertainty is the first order error propagation proposed in [16]. The optimization problem

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that is considered is not weighted and therefore identical isotropic noise is assumed for all the points. The author defines a first order error model that is propagated through the solution, while assuming independent and identically distributed points (no longer necessarily isotropic). It is noted that the findings of the aforementioned works are all restricted to three-dimensional points. In [17] a different optimization problem is proposed that accounts for independent anisotropic noise affecting rotated-only point sets also in three dimensions. The authors determine the theoretical lower bound for the covariance of the rotation error in that case, and through an iterative solution recurring to quaternion representation reach the theoretical bound. Besides the iterative solution, some shortcomings of this work are its limitation to the tridimensional problem with rotation-only, and the fact that, although anisotropic, the input covariances are normalized and share a common normalizing factor. Regarding the stability of the solution, Söderkvist [18] addresses the study of this issue when the algorithm is exposed to perturbed data sets, concluding that the singular values of the matrices composed with the points in each set are closely related to the conditioning of the problem, while finding a bound for the perturbation on the rotation matrix when the input perturbations are bounded. In related directions of research that demonstrate the relevance of pattern point matching, and, consequently, of point registration problems such as the Procrustes problem, the authors of [19,20] propose algorithms that exploit different approaches to registration and matching. Furthermore, the latter is an iterative algorithm that, assuming a rotation–scaling–translation transformation between two sets of points, finds the point correspondences and a variational Bayesian approximation for the distribution of the transformation.

This paper addresses the n-dimensional (n-D) extended orthogonal Procrustes problem considering a transformation composed of a rotation and a translation (no scaling). The problem is posed with individual scalar weights for each pair of points, and a closed-form solution is presented. Data association is assumed to be performed a priori. Founded on perturbation theory, a novel and general uncertainty description for the solution of the optimization problem is proposed. Building on the results presented in [13,14,16], and assuming a stochastic perturbation model for the point sets with individual covariance matrices for each point, as well as cross-covariances for each pair of points, the first and second moments of the resulting translation and rotation are computed. This is achieved considering arbitrary rotations and translations, individual weights, and full covariance matrices for both point sets. As a by-product of this work, an application to robotics was proposed in [21,22] within the scope of simultaneous localization and mapping [23]. In this application, if a landmark map (or set of points) is available in a coordinate frame attached to the robot, it is possible to compute the transformation between that frame and another frame fixed to the initial position of the robot. Following this idea, an online Earth-fixed trajectory and map estimation algorithm based on the Procrustes problem was proposed and its uncertainty characterization derived, making full use of the methodology proposed in this paper. This builds on the previous works by the authors, where globally asymptotically stable filters for simultaneous localization and mapping in a sensor-based or robocentric framework were proposed for bidimensional [24] and tridimensional mission scenarios [25]. The performance and consistency of the overall algorithm are validated in a real world environment for both dimensionalities, showing that the algorithm provides accurate and consistent estimates, and, therefore, also providing an experimental validation of the uncertainty characterization proposed in this paper.

The contributions of this paper are: (i) the full uncertainty characterization of the optimization problem of obtaining the transformation between corresponding n-dimensional point sets and its closed-form solution, while considering point sets perturbed by anisotropic noise, and points that are not required to be independent nor identically distributed; and (ii) a thorough validation of the uncertainty characterization, using extensive Monte Carlo simulations to study the main properties of the proposed methodology. This paper builds on the preliminary versions of this work presented in [21,22], by reformulating the problem of obtaining the pose of the vehicle, while extending the derivation therein to points of arbitrary dimensions. In contrast with the latter work, this paper provides new theoretical results, generalizes the proposed uncertainty characterization to $\mathbb{R}^n$, and provides statistical validation through extensive Monte Carlo simulations for several dimensions and a multitude of parameter combinations.

The applications of Procrustes analysis are found in a wide variety of fields, which can benefit from the proposed approach, including rigid body motion, vibration tests of large complex structures [26], structural and system identification, factor analysis in n-D (e.g. checking whether two matrices are equivalent), similarity evaluation in statistical data sets [27, Chapter 20], medical imaging [14], photogrammetry [28], shape comparison (generalized Procrustes analysis) [29], and quantitative psychology [30] (where the problem was initially solved). In recent years several algorithms were developed in the field of computer vision that availed themselves of the Procrustes problem, from shape matching and retrieval [31] to similarity search in image collections [32], among others. Shape matching is in fact a more complex problem, as the problem of finding the transformations is coupled with the problem of finding the reference shape to which all the measured shapes relate. In [33] the authors propose a unifying framework that has a closed-form computation for affine, similarity or Euclidean transformations between a set of shapes, while allowing us to find the underlying shape and accounting for missing pairs of points. All this is performed considering noise in the measured shapes and not in the reference-space as is customary. Other applications include non-negative matrix factorization [34], and phase FIR filter bank design [35], whereas the work in [36] underlines the importance of addressing the problem in less common dimensionalities, such as four-dimensional shapes. Another possible application of the Procrustes problem lies in iterative closest point algorithms such as [37], even though most use quaternions to parametrize the rotation of the sets. If the registration is performed in each step with a constrained least squares approach, it can benefit from the characterization here proposed. Another interesting application of Procrustes analysis is manifold alignment [38] in the area of machine learning. In this n-dimensional technique, it is argued that it is possible to model the underlying structure of most datasets by manifolds, whose alignment then allows for knowledge transfer across datasets. The authors of [38] demonstrate the validity of this approach by applying the idea to learning transfer in reinforcement learning with Markov Decision Processes, alignment of the tertiary structure of proteins, cross lingual information alignment, within others. These demonstrate the real world relevance of the n-dimensional Procrustes problem in several fields even for dimensionalities outside of the 2-D/3-D common problems. Furthermore, given the noisy nature of these problems, the proposed uncertainty characterization can be useful to compute the reliability of the alignment resulting from the Procrustes procedure.

Paper structure: The paper is organized as follows. Section 2 presents a brief overview of some mathematical concepts needed in the course of this paper. Section 3 presents the formulation and closed-form solution of the weighted orthogonal Procrustes problem. A novel uncertainty characterization of this problem is derived in Section 4 and validated in Section 5 through extensive