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Lagrangian relaxation graph matching ☆

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ABSTRACT

Graph matching is a fundamental problem in computer vision area. Graph matching problem that incorporates pair-wise constraints can be formulated as an integer quadratic programming (IQP) problem with affine mapping constraint. Since it is known to be NP-hard, approximate relaxation methods are usually required to find approximate solutions. In this paper, we present a new effective graph matching relaxation method, called Lagrangian relaxation graph matching (LRGM), which aims to generate a relaxation model by incorporating the affine mapping constraint into the matching objective optimization. There are three main benefits of the proposed LRGM method: (1) The nonnegative affine mapping constraint encoding one-to-one mapping is naturally incorporated in LRGM relaxation via Lagrangian regularization. (2) By further adding a ℓ_1-norm constraint, LRGM can generate a sparse solution empirically and thus returns a desired discrete solution for original IQP matching problem. (3) An effective update algorithm is derived to solve the proposed LRGM model. Theoretically, the converged solution can be proven to be Karush–Kuhn–Tucker (KKT) optimal. Experimental results on both synthetic data and real-world image datasets show the effectiveness and benefits of the proposed LRGM method.

1. Introduction

Many problems in computer vision in pattern recognition area can be solved by graph matching methods [1–7]. Graph matching problem that incorporates pairwise constraints can be formulated as an integer quadratic programming (IQP) problem with affine mapping constraint encoding one-to-one mapping constraint. The optimal solution of IQP matching problem should satisfy both discrete and affine mapping constraints simultaneously. Since it is known to be NP-hard, relaxation methods are usually required to find approximate solutions for this problem [8,6,9–12]. One kind of popular relaxation methods is to first develop a new continuous problem by relaxing the discrete mapping constraint and then aim to find the optimum for this relaxed continuous problem. Since the optimum of the relaxed matching problem is usually continuous, thus a post-optimization step (discretization step) is further required to obtain the final discrete (binary) mapping solution [9–11,13]. One main limitation is that the required post-optimization step is generally independent of the matching objective optimization and thus may lead to weak local optimum for original IQP matching problem. Another kind of relaxation method is to try to optimize IQP matching problem in a discrete domain [14]. This method can generate a solution obeying the discrete affine mapping constraint strictly for original IQP matching problem, and thus does not require any post-optimization step in the optimization process [14]. However, the optimality of this discrete method is usually limited. Zhou and la Torre [12] proposed an effective graph matching method which first exploited the factorized properties of the affinity matrix and then optimized the factorized graph matching problem using a path-following algorithm [15]. This method is usually time consuming because it needs to solve a series of sub-problems. From the game-theoretical perspective, Albarelli et al. [16,17] have proposed a new relaxation model which aims to optimize the matching problem in a simplex domain, i.e., ℓ_1-norm constraint in nonnegative domain. For convenience, we call it as GameM in the following. Due to the ℓ_1-norm constraint, GameM can induce a sparse (approximate discrete) solution for the matching problem [17]. One drawback is that the mapping constraint encoding one-to-one mapping has been entirely ignored in GameM. Also, the sparsity of GameM solution is usually uncontrollable to obtain the desired discrete solution [17,18]. In addition to optimization-based methods, probabilistic methods can also be used for solving matching problems [8,3].

In this paper, a new graph matching method has been proposed. The main contributions are twofold: First, a new relaxation model, called Lagrangian relaxation graph matching (LRGM), has been proposed for solving IQP matching problem. LRGM has two aspects: (1) the affine mapping constraint is explicitly

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incorporated in LRGM relaxation via Lagrangian regularization. (2) By imposing ℓ_1-norm constraint on related solution, empirically LRGM can also lead to a sparse solution and thus maintains the discrete constraint strongly. Thus, LRGM provides a tight relaxation intuitively for original IQP problem by incorporating both affine mapping and discrete binary constraint simultaneously. Second, an effective multiplicative optimization algorithm has been proposed to solve LRGM model. The convergence of the algorithm is guaranteed. Another important property of this algorithm is that it leads to an optimal solution which can satisfy the Karush–Kuhn–Tucker (KKT) optimal condition.

The remainder of this paper is organized as follows. In Section 2, we introduce the general formulation of graph matching problem as an integer quadratic programming problem. In Section 3, we briefly review some related works. In Section 4, we propose our LRGM model and develop a multiplicative update algorithm to solve it. Some benefits of the proposed LRGM method are demonstrated in Section 5. At last, we apply LRGM method to some matching tasks.

2. Problem formulation

Assume that two attributed graphs to be matched are $G^D = (V^D, E^D, A^D, R^D)$ and $G^M = (V^M, E^M, A^M, R^M)$ where V represents a set of nodes, E, edges, and A, unary attributes, R, binary relations. Each node $v_i^D \in V^D$ or edge $e_{ik}^D \in E^D$ has an associated attribute vector $\mathbf{a}_i^D \in A^D$ or $\mathbf{r}_{ik}^D \in R^D$. For example, in image feature matching task, the attribute vector $\mathbf{a}_i^D$ can be used to represent some feature descriptors, such as SIFT descriptor and color histogram, and binary relation $\mathbf{r}_{ik}^D$ usually refers to the pairwise relationship between features, such as distance. The aim of graph matching problem is to determine the correct correspondences between V^D and V^M. A correspondence mapping is a set of pairs (or assignments) (v_i^D, v_j^M). For each assignment pair (v_i^D, v_j^M) and (v_k^D, v_l^M), there is an affinity $\mathbf{W}_{ij,kl} = f_r(\mathbf{r}_{ik}^D, \mathbf{r}_{jl}^M)$ that measures how compatible the nodes (v_i^D, v_k^D) in graph G^D are with the nodes (v_j^M, v_l^M) in graph G^M. Thus, we can use a matrix $\mathbf{W}$ in which the non-diagonal element $\mathbf{W}_{ij,kl}$ contains a pair-wise affinity between two assignments (v_i^D, v_j^M) and (v_k^D, v_l^M), and the diagonal term $\mathbf{W}_{ij,ij}$ represents an unary affinity of correspondence (v_i^D, v_j^M). The correspondences between two graphs can be represented by a permutation matrix $\mathbf{X}$, $\mathbf{X} \in \mathbb{R}^{m\times n}(m = |V^D|, n = |V^M|)$, where $\mathbf{X}_{ij} = 1$ implies that node v_i^D in G^D corresponds to node v_j^M in G^M, and $\mathbf{X}_{ij} = 0$ otherwise. In this paper, we denote $\mathbf{x} \in \{0, 1\}^{mn}$ as a row-wise vectorized replica of $\mathbf{X}$, i.e., $\mathbf{x} = \mathrm{vec}(\mathbf{X}) = (\mathbf{X}_{11}\ldots\mathbf{X}_{1n}, \ldots, \mathbf{X}_{m1}\ldots\mathbf{X}_{mn})^{\mathrm{T}}$. In the following, $\mathbf{X}$ is called as the matrix form of $\mathbf{x}$, i.e., $\mathbf{X} = \mathrm{mat}(\mathbf{x})$, and $\mathbf{x}$ is called as the vector form of $\mathbf{X}$, i.e., $\mathbf{x} = \mathrm{vec}(\mathbf{X})$. The graph matching problem, in its most recent and general form, can be formulated as an integer quadratic programming (IQP) problem, i.e., finding the indicator vector $\bar{\mathbf{x}}$ that maximizes the following score function [14,8,9,19],

$$\max_{\mathbf{x}} \quad \sum_{ij}\sum_{kl} \mathbf{W}_{ij,kl}\mathbf{X}_{ij}\mathbf{X}_{kl} = \mathbf{x}^{\mathrm{T}}\mathbf{W}\mathbf{x} \tag{1}$$

$$\text{s. t.} \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x}_i \in \{0, 1\}. \tag{2}$$

where $\mathbf{A} \in R^{(m+n)\times mn}$ and $\mathbf{b} \in R^{(m+n)\times 1}$ are set to encode the doubly stochastic constraint, i.e.,

$$\mathbf{A}_{ij} = \begin{cases} 1 & \text{if } i, j \in S_r \bigcup S_c \\ 0 & \text{otherwise} \end{cases}$$

where $S_r = \{i, j | i \le m, n(i-1) + 1 \le j \le n \times i\}$ and $S_c = \{i, j | m < i \le m + n, j \in \{i - m, i - m + n, \cdots i - m + (m-1)n\}\}$. $\mathbf{b} = (1, 1, \cdots 1)^{\mathrm{T}} \in R^{(m+n)\times 1}$. The affine constraint $\mathbf{Ax} = \mathbf{b}$[1] ensures one-to-one mapping constraint between V^D and V^M [8,19], which is mostly concerned in many matching tasks [8,19,12,14].

3. Related works

It is known that the above IQP matching problem is NP-hard, thus approximate relaxations are usually required. Here, we briefly review some popular models that are closely related to our work.

Spectral matching (*SM*): By relaxing both integer and affine mapping constraint, Leordeanu and Hebert [10] proposed a relaxed problem as follows:

$$\max_{\mathbf{x}} \quad \mathbf{x}^{\mathrm{T}}\mathbf{W}\mathbf{x}$$
$$\text{s. t.} \quad \| \mathbf{x} \|_2 = 1, \tag{3}$$

where $\| \mathbf{x} \|_2 = (\sum_i \mathbf{x}_i^2)^{1/2}$. SM has a closed-form solution which is the leading eigenvector of the affinity matrix $\mathbf{W}$. However, the matching constraints involving both discrete constraint $\mathbf{x}_i \in \{0, 1\}$ and affine constraint $\mathbf{Ax} = \mathbf{b}$ are ignored in this relaxed model. Therefore, the optimal solution of SM model is continuous and should be further discretized to obtain the final discrete mapping solution.

Spectral matching with affine constraints (*SMAC*): By further adding the affine mapping constraint to SM relaxation, Cour et al. [9] proposed a relaxation as follows:

$$\max_{\mathbf{x}} \quad \mathbf{x}^{\mathrm{T}}\mathbf{W}\mathbf{x}$$
$$\text{s. t.} \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \quad \| \mathbf{x} \|_2 = 1. \tag{4}$$

Comparing with SM, SMAC provides a tighter relaxation and also has a closed-form solution. However, similar to SM, the discrete constraint $\mathbf{x}_i \in \{0, 1\}$ is also dropped in SMAC and thus should be further discretized to obtain the final discrete solution.

Game-theoretic matching (*GameM*): From game-theoretic perspective, Albarelli et al. [17] have proposed a relaxed matching model by replacing the ℓ_2 norm constraint in SM with ℓ_1 norm, i.e.,

$$\max_{\mathbf{x}} \quad \mathbf{x}^{\mathrm{T}}\mathbf{W}\mathbf{x}$$
$$\text{s. t.} \quad \| \mathbf{x} \|_1 = 1, \quad \mathbf{x}_i \ge 0 \tag{5}$$

where $\| \mathbf{x} \|_1 = \sum_i |\mathbf{x}_i|$. Comparing with SM (Eq. (3)), one important feature of GameM is that it can generate a sparse solution for the matching problem due to ℓ_1 norm constraint, and thus incorporates the desirable discrete binary constraint approximately in optimization. However, one drawback is that the affine matching constraint encoding one-to-one mapping is entirely ignored in GameM.

Our aim in this paper is to propose a new relaxation for the IQP matching problem (Eq. (1)). We call it as Lagrangian relaxation graph matching (LRGM). The main benefit of LRGM is that it integrates both affine and discrete mapping constraints of IQP matching problem (Eq. (1)) in relaxation process and thus integrates the benefits of SMAC and GameM simultaneously.

[1] Generally, $\mathbf{Ax} \le \mathbf{b}$ is used for graphs with different sizes. Here, we focus on equal-size graph matching problem. For the graphs with different sizes, one can transform them to the same size by adding some dummy isolated nodes into the smaller graph [12,15].