Journal of Quantitative Spectroscopy & Radiative Transfer 204 (2018) 206–214

Contents lists available at ScienceDirect

Journal of Quantitative Spectroscopy & Radiative Transfer

journal homepage: www.elsevier.com/locate/jqsrt

Accurate expansion of cylindrical paraxial waves for its straightforward implementation in electromagnetic scattering

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ARTICLE INFO

Article history:
Received 24 May 2017
Revised 25 September 2017
Accepted 25 September 2017
Available online 27 September 2017

Keywords:
Focal waves
Diffraction
Wave expansion

ABSTRACT

The evaluation of vector wave fields can be accurately performed by means of diffraction integrals, differential equations and also series expansions. In this paper, a Bessel series expansion which basis relies on the exact solution of the Helmholtz equation in cylindrical coordinates is theoretically developed for the straightforward yet accurate description of low-numerical-aperture focal waves. The validity of this approach is confirmed by explicit application to Gaussian beams and apertured focused fields in the paraxial regime. Finally we discuss how our procedure can be favorably implemented in scattering problems.

1. Introduction

The description of unbounded wave fields in the form of series expansion represents a widely used tool in free space propagation [1–3]. For instance, the electric field emerging from a laser device is commonly expressed as a combination of either Hermite–Gauss or Laguerre–Gauss basis functions taken from the natural modes existing in mirror cavities [4–6]. Such procedure, however, seems inappropriate for apertured focal waves where diffraction sidelobes will impose a strong limitation in its efficient characterization by an acceptable truncation of the sequence [7,8]. In these cases, expansions in terms of Lommel functions demonstrate a suitable implementation [9,10].

In planar arrangements, two-dimensional (2D) waves satisfying the Helmholtz equation can be described in terms of Bessel wave functions when they are formulated in cylindrical coordinates [11]. This is a method followed for instance in the Mie–Lorenz theory applied to scatterers with circular cross section [12–14]. Only a small part of the cases analyzed in scattering problems consider non-uniform beams, among other reasons due to the complexity of the series expansions with cylindrical vector wave functions (also spherical vector wave functions) that can be found [15–20]. Particularly interesting is the paper from Shogo Kozaki which analyzes the case of a Gaussian beam illuminating a cylindrical particle, where it is possible to remarkably simplify the analytical description of the beam, under certain approximations, in terms of Bessel cylindrical waves [21].

A salient feature of these Bessel wave functions is their inherent localization around the chosen origin of coordinates. Therefore, focal waves are generally expected to be favorable candidates to be effectively represented by means of a series expansion using Bessel wave fields.

In this paper, we consider 2D paraxial wave fields which are localized in the vicinity of a given point, which serves for the construction of the Bessel wave-functions basis to be used in a series expansion. The coefficients of such sequence are estimated by analyzing the far field of the paraxial wave. This choice reduces the resultant calculation to a simple Fourier expansion. The validity of this method is finally verified when applied to a Gaussian laser beam and also to an apertured focal wave.

2. Theoretical analysis

Let us consider a 2D wave field $E(x, z) = \exp(ikz)U(x, z)$ propagating in free space and satisfying the paraxial wave equation, $\partial_x^2 U + 2ik\partial_z U = 0$, where z represents the spatial coordinate along the optical axis, x is the transverse coordinate and $k = 2\pi/\lambda$ is the wavenumber. Such scalar description of the wave field is fully satisfactory provided that the electric field is oriented along the y-axis, as we will consider here unless otherwise indicated; note that the duality theorem enables the use of a scalar wave field also in the case that the electric field lies on the xz plane [11]. Note that a harmonic time variation $\exp(-i\omega t)$ is here assumed, where ω is the time-domain frequency. If the focal point is set at the origin of coordinates, $(x, z) = (0, 0)$, the wave field at any out-of-focus plane

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https://doi.org/10.1016/j.jqsrt.2017.09.023
0022-4073/

$z \neq 0$ can be determined by means of the Fresnel diffraction integral [22]

$$E(x,z) = \frac{\exp(ikz)}{\sqrt{i\lambda z}} \int_{-\infty}^{\infty} E(x_0,0) \exp\left[\frac{ik}{2z}(x-x_0)^2\right] dx_0. \tag{1}$$

The characteristic Fraunhofer pattern can be observed well beyond the focal plane in the limit $|z| \to \infty$. In this case, the quadratic phase term $\exp\left(ikx_0^2/2z\right)$ in the integral Eq. (1) is negligible.

When $z > 0$, the far field of $E(x, z)$ can be expressed as

$$E(x,z) \to \frac{\exp(ikr)}{\sqrt{r}} \exp(-i\pi/4)\sqrt{\lambda} a_F(\theta), \tag{2}$$

where the angular spectrum

$$a_F(\theta) = \frac{1}{\lambda} \int_{-\infty}^{\infty} E(x_0,0) \exp(-ik\theta x_0) dx_0, \tag{3}$$

is simply the Fourier transform of the wave field at the focal plane. We point out that Eq. (2) rigorously describes the paraxial wave field only in the limit $z \to \infty$, but it provides accurate results at a distance sufficiently far from focus. The azimuthal angle measured from the optical axis is given as $\theta = x/z$ in the paraxial approximation, whereas the radial coordinate is given by $r = |z + x^2/2z|$. In the paraxial regime, the wave function $a(\theta)$ has significant values only when $|\theta| \ll 1$.

Now it is clear that a wave expansion can be given by means of a Fourier series of the angular spectrum, that is

$$a_F(\theta) = \sum_{n=-\infty}^{\infty} a_n \exp(in\theta), \tag{4}$$

where the Fourier coefficient

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} a_F(\theta) \exp(-in\theta) d\theta. \tag{5}$$

Since the angular spectrum $a(\theta)$ takes values identically zero in the range $|\theta| > \pi/2$ in the semi-space $z > 0$, the definite integral given in Eq. (5) can be further simplified by extending the interval of integration to $-\infty < \theta < \infty$. Under such approximation, and substituting Eq. (3) into (5), we finally infer that the Fourier coefficient of order n

$$a_n = \frac{1}{2\pi} E\left(-\frac{n}{k}, 0\right), \tag{6}$$

depends on the focal field as measured at the off-axis point $x = -n/k$.

To extend the series expansion of the wave field into the near field (note that here we disregard evanescent waves and near field refers to waves in the vicinities of focus), we will use the Bessel wave functions of the first kind and order n, $J_n(kr)$, and the Bessel functions of the second kind and order n, $Y_n(kr)$, which are solutions of the 2D Helmholtz wave equation, $\nabla^2 E + k^2 E = 0$, provided that the angular variation of the wave field is given by $\exp(in\theta)$ [11]. In particular, the Hankel wave function of the first kind $H_n^{(1)}(kr) = J_n(kr) + iY_n(kr)$ has an asymptotic limit far from the origin of coordinates given as

$$H_n^{(1)}(kr) \to \sqrt{\frac{2}{\pi kr}} \exp(ikr) \exp(-in\pi/2) \exp(-i\pi/4). \tag{7}$$

Therefore, an outgoing cylindrical wave field with a specific angular momentum can be given in terms of a Hankel wave function of the first kind and unique order n. As a consequence, a wave field exhibiting an angular spectrum $a_F(\theta)$ in the semi-space $z > 0$ can be expressed as a combination of Hankel wave functions of different orders n as

$$E_F^+(x,z) = \pi \sum_{n=-\infty}^{\infty} a_n \exp(in\pi/2) H_n^{(1)}(kr) \exp(in\theta), \tag{8}$$

which in addition can be utilized in the near field.

Eq. (8) provides the expression of a wave field whose Fraunhofer pattern is given in terms of the wave function $a_F(\theta)$, leading to a vanishing far field in $z < 0$. However, in such semi-space, the far field of $E(x, z)$ given in Eq. (1) can be expressed as

$$E(x,z) \to \frac{\exp(-ikr)}{\sqrt{r}} \exp(i\pi/4)\sqrt{\lambda} a_F(\theta'), \tag{9}$$

in the limit $z \to -\infty$, where the angle $\theta' = x/z$ in the paraxial approximation. With the new angular coordinate $\theta' = \theta - \pi$, we again may describe the Fraunhofer pattern observed at $z \to -\infty$ by means of the wave function $a_F(\theta')$ which takes significant values for $|\theta'| \ll 1$. Furthermore, the Fourier series given in Eqs. (4) and (5) are still applicable in this case. Taking into account the asymptotic limit of the Hankel wave function of the second kind $H_n^{(2)}(kr) = J_n(kr) - iY_n(kr)$, written as

$$H_n^{(2)}(kr) \to \sqrt{\frac{2}{\pi kr}} \exp(-ikr) \exp(in\pi/2) \exp(i\pi/4), \tag{10}$$

we may infer a series expansion of a wave field with angular spectrum $a_F(\theta')$ in terms of incoming cylindrical waves, resulting in

$$E_F^-(x,z) = \pi \sum_{n=-\infty}^{\infty} a_n \exp(-in\pi/2) H_n^{(2)}(kr) \exp(in\theta'), \tag{11}$$

which is also accurate in the near field.

The appropriate description of the paraxial wave field $E(x, z)$ includes a far field at $z \to \infty$ in the form of an outgoing cylindrical wave patterned by $a_F(\theta)$, and simultaneously at $z \to -\infty$ representing an ingoing cylindrical wave shaped by the same angular spectrum $a_F(\theta')$. Note that such symmetry around the focal point has been analyzed by Collet and Wolf [23]. As a consequence, the wave field must be computed as $E(x,z) = E_F^+(x,z) + E_F^-(x,z)$, which in cylindrical coordinates is expressed as

$$E(r,\theta) = 2\pi \sum_{n=-\infty}^{\infty} i^n a_n J_n(kr) \exp(in\theta). \tag{12}$$

This is the main result of our study, together with the fact that the Fourier coefficients as given in Eq. (6) can be achieved in terms of the focal wave field. We conclude that a cylindrical, paraxial wave field can be accurately described by a series expansion sustained by Bessel wave functions which are solutions of the 2D Helmholtz equation, and with expansion coefficients which are determined by the own wave field at specific points of the focal plane. We point out that Eq. (12) accurately provides the wave field even at angles far from the paraxial regime where $\tan\theta = x/z$; of course $r = \sqrt{x^2 + z^2}$ denotes the distance from focus to the observation point.

In centrosymmetric field distributions where $E(-x,z) = E(x,z)$, what occurs simply if the field is symmetric with respect to the origin of coordinates at the focal plane, a reduced expression of the series expansion can be deduced. In this case, $a_{-n} = a_n$ as inferred from Eq. (6). By using the following property of Bessel functions of negative order, $J_{-n}(\alpha) = (-1)^n J_n(\alpha)$, we finally may reduce Eq. (12) to

$$E(r,\theta) = E_0 J_0(kr) + 4\pi \sum_{n=1}^{\infty} i^n a_n J_n(kr) \cos(n\theta), \tag{13}$$

where $E_0 = 2\pi a_0$ is the in-focus wave field.

3. Implementation in scattering problems

Let us consider a scatterer embodied in a cylindrical region whose axis is set along the y axis and with a radius R, as depicted in Fig. 1. In order to analytically estimate the scattered wave