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The effect of bifurcation on numerical calculation of conjugate heat transfer with radiation ☆

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ABSTRACT

The behavior of numerical solutions to conjugate heat transfer problems when thermal radiation is significant is discussed. Hidden behavior that can prevent convergence of numerical techniques is shown through a simple example and comparison with analytical solution of the resulting quartic equation. The paper illustrates why the nonlinear form of the governing energy equations can present unexpected behavior in numerical solutions, and this can prevent converged solutions in many cases. Discussion of whether apparent bifurcation/chaos in the solution has meaning in this class of problems is discussed.

1. Introduction

Attention to the observed characteristics of nonlinear equations and the bifurcation/chaos maps of their solution leads to the question of whether such effects occur in standard conjugate heat transfer problems. It is well-known that the highly nonlinear equations describing multi-mode heat transfer problems (particularly when radiation is important) almost invariably require numerical solution. It is less well recognized that under certain conditions this nonlinear nature can lead to numerical solutions that exhibit the characteristics of bifurcation and chaos, even though such characteristics are not expected in the physical behavior of the systems being modeled (excepting for some problems involving natural convection, when the physical behavior can indeed involve chaotic characteristics).

This paper illustrates this numerical behavior, and serves to help identify the problems that can result in using numerical approaches to physical solutions described by nonlinear equations. This is done by choosing a very simple example problem and showing the numerical behavior present in the solution, observing that such behavior is probably present in the entire range of multi-mode problems when radiation is important, not just in the simple case presented here.

2. Analysis

Consider the one-dimensional problem shown in Fig. 1, where energy is transferred by conduction from a surface at T_0 through the thin wall on the left to surface 1, and then by radiation through the transparent medium between black surfaces 1 and 2.

The energy equation describing this problem is

$$q = \frac{k}{L}(T_0 - T_1) = \sigma\left(T_1^4 - T_2^4\right) \quad (1)$$

Let $\theta = T/T_0$; $C = \left(k/\sigma T_0^3 L\right)$; $D = \left(\theta_2^4 + C\right)$. Substituting, Eq. (1) becomes

$$\theta_1^4 + C\theta_1 - D = 0 \quad (2)$$

Any general quartic equation of the form $\theta^4 + \alpha\theta^3 + \beta\theta^2 + \gamma\theta + \delta = 0$ has an analytical solution [e.g., Abramowitz and Stegun [1], pg 17], and Eq. (2) is a special case with $\alpha = \beta = 0$. Various approaches to solution of such quartic equations have been used in the past. These are discussed at length by Shmakov [2], who also covers the history of the various methods and derives a general solution formulation. All of the methods are based on finding the roots of a related cubic equation (the resolvent cubic), which are then used to find the roots of the quartic. Because all of the physical factors making up the coefficients C and D in Eq. (2) are positive and real, the four roots of the quartic include only one positive real root, two imaginary roots and one negative real root. As the value of the dimensionless absolute temperature θ_1 for the posed problem must be real and positive, only the positive real

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Nomenclature

C	$k/\sigma T_0^3 L$
D	$\theta_2^4 + C$
k	Thermal conductivity
L	Thickness of conducting layer
q	Energy rate/area
Q, S	Intermediate relations for quartic solution, Eq. 3
T	Absolute temperature
θ	T/T_0
σ	Stefan-Boltzmann constant

Subscripts

0, 1, 2	Surface 0, 1, or 2.
i	Iteration index

root need be found and considered. This root may be found from:

$$\theta_1 = -S + \left(\frac{1}{2}\right)\left(-4S^2 + \frac{C}{S}\right)^{1/2} \tag{3}$$

where $S = \left(\frac{1}{2}\right)\left[\frac{1}{3}\left(Q - \frac{12D}{Q}\right)\right]^{1/2}$; $Q = \left(\left(\frac{1}{2}\right)\left\{27C^2 + \left[\left(27C^2\right)^2 + 4(12D)^3\right]^{1/2}\right\}\right)^{1/3}$.

Most quartic equations describing multi-mode problems (especially those involving radiatively participating media) are considerably more complex than Eq. (2), and may in general be integro-differential equations. Analytical solutions are then not usually available. Numerical methods must be invoked to find a solution. The behavior of these more complicated and still non-linear relations still have embedded behavior that parallels that found for the simple example.

3. Numerical solutions and their pitfalls

Suppose it is not known that the analytical solution to Eq. (2) exists, and a numerical solution is resorted to. The usual approach is to invoke successive approximation, and Eq. (2) might be re-written in the form

$$\theta_{1,i+1} = \left(\frac{1}{C}\right)\left(D - \theta_{1,i}^4\right) \quad i = 0, 1, 2, 3, \ldots \tag{4}$$

where the additional subscript i now denotes the iteration number on the unknown θ_1. This form is preferred if $\theta_1{}^4$ << D (radiation is a small effect). If convection is small compared with radiation, then a better form for achieving convergence might be

$$\theta_{1,i+1}^4 = D - C\theta_{1,i} \quad i = 0, 1, 2, 3, \ldots \tag{5}$$

To indicate the numerical problem that can arise, take one simple case. Suppose the values of the parameters describing the problem posed in Fig. 1 are found to be $H = 4$, $K = 1$, and $\theta_2 = 0.5$ ($C = 5$, $D = 3.063$). Given these values, we want to find the dimensionless temperature θ_1 a t surface 1. The analytical solution from Eq. (3) to the resulting quartic equation of $\theta_1^4 + 5\theta_1 - 3.063 = 0$ is found to be $\theta_1 = 0.589$, and the numerical solution using Eq. (4) is found (as expected) to be the same. Eq. 4 converges very quickly for this combination of C and D (to within six significant figures in 8 iterations) for any reasonable initial value θ_1 (*reasonable* meaning that it must be in the range $0.5 < \theta_1 < 1$ as constrained by the boundary conditions).

Suppose we now examine a range of values for the parameter H, reducing it so that the effect of convection becomes smaller. The graph in Fig. 2 shows the analytical solution, and down to a value of about $H = 0.4436$, the numerical solution tracks the analytical solution exactly, although the number of iterations required for convergence climbs rapidly as H is reduced in value. At $H = 0.450$, over 1800 iterations are required to converge to six significant figures. At $H = 0.4436$, convergence is not achieved even at 20,000 iterations (where the last two successive iterations have produced values of 0.71026 and 0.71345), although the procedure continues to slowly approach a converged value of 0.71186.

Below a value of $H = 0.4434$, the numerical solution shows bifurcation. When a value of $\theta_1{}^*$ is inserted into the RHS of Eq. (4), a value $\theta_1{}^{**}$ is returned that, when input into the RHS in the next iteration, returns the previous value $\theta_1{}^*$. Convergence is thus not possible using simple successive approximation. The map of this effect is shown in Fig. 3, where the two cyclic values are shown by the diverging wings at values of H below 0.442. For example, at $H = 0.4$, the two cyclic values are $\theta_1{}^* = 0.5616$ and $\theta_1{}^{**} = 0.8307$. The dashed line shows the continuing solution provided by the

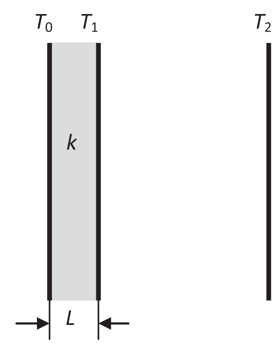

Fig. 1. Geometry for Example Multimode Problem: T_0 and T_2 known, find T_1.

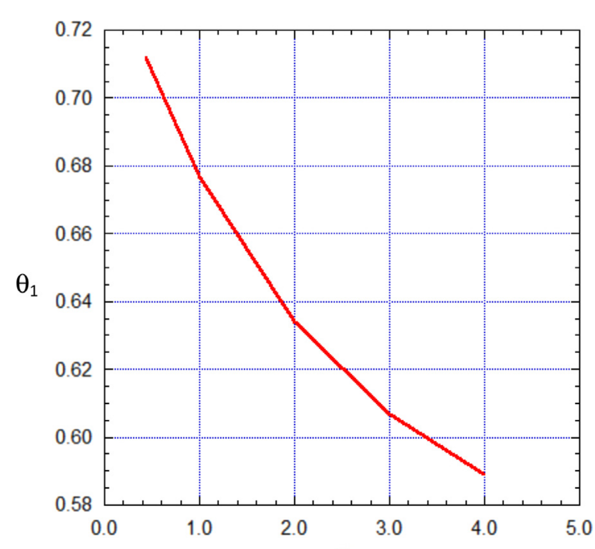

Fig. 2. Temperature of Surface 1 for Parameters $K = 1$, $\theta_2 = 0.5$ from Both Numerical and Analytical Solutions of Eq. 2.