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A class of upper and lower triangular splitting iteration methods for image restoration

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ABSTRACT

Based on the augmented linear system, a class of upper and lower triangular (ULT) splitting iteration methods are established for solving the linear systems arising from image restoration problem. The convergence analysis of the ULT methods is presented for image restoration problem. Moreover, the optimal iteration parameters which minimize the spectral radius of the iteration matrix of these ULT methods and corresponding convergence factors for some special cases are given. In addition, numerical examples from image restoration are employed to validate the theoretical analysis and examine the effectiveness and competitiveness of the proposed methods. Experimental results show that these ULT methods considerably outperform the newly developed methods such as SHSS and RGHSS methods in terms of the numerical performance and image recovering quality. Finally, the SOR acceleration scheme for the ULT iteration method is discussed.

1. Introduction

Images are usually degraded by blur and noise during image acquisition and transmission. Image restoration is a widely studied problem in several applied scientific areas, such as removing the noise from magnetic resonance images (MRIs), chest X-rays, and digital angiographic images in medical imaging [1,2], restoration of aging and deteriorated films in engineering [3], restoring degraded images obtained by telescopes or satellites in astronomy [4], restoration of degraded images in optical systems [5], and many other areas (see [6,7]). Restoration is a process that involves reconstructing or recovering a degraded image using a priori knowledge related to the degradation phenomenon. The input–output relationship of image restoration can be written as follows [8]:

$$g(x,y) = \mathbb{H}[f(x,y)] + n(x,y), \tag{1.1}$$

where $\mathbb{H}$ is a degradation operator, $f(x, y)$ is the original image, $g(x, y)$ is the degraded image (recorded image), and $n(x, y)$ is additive noise. It can be shown that if $\mathbb{H}$ is a linear and space-invariant operator, then Eq. (1.1) can be written as the first type Fredholm integral equation

$$g(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(x-\xi, y-\eta) f(\xi,\eta) d\xi d\eta + n(x,y), \tag{1.2}$$

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where $h(x, y)$ is usually known as the point spread function (PSF) and $n(x, y)$ is independent of the spatial coordinates. In this paper, the PSF is assumed to be known. The discretization of (1.2) leads to the following formulation:

$$g(x,y) = \sum_{k=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty} h(x-k, y-l)f(k,l) + n(x,y). \tag{1.3}$$

It has been shown that the Eq. (1.3) can be expressed in the matrix–vector equation as [8]

$$g = Af + \eta, \tag{1.4}$$

where A is a blurring matrix of size $n^2 \times n^2$ and f, η and g are n^2-dimensional vectors representing the original image, noise, and blurred and noisy (degraded) image, respectively. Given some assumptions of the value outside the field of view (FOV) are known as boundary conditions (BCs). The structure of the blurring matrix A depends on the used BCs. Zero, periodic, reflexive, antireflective and mean are five known BCs which have been widely used in the literature. In the zero BCs, it is assumed that the outside of FOV is zero (black). The proposed assumption leads to block Toeplitz with Toeplitz blocks (BTTB) for blurring matrix A. The periodic BCs is implemented by considering the periodic extension of data in the outside of FOV. In this BCs, the matrix A has the block circulant with circulant blocks (BCCB) structure. It is shown that the matrix–vector multiplications are effectively computed by fast Fourier transforms (FFTs) in zero and periodic BCs [7]. Reflecting the FOV data to outside leads to the reflexive BCs. In this BCs, the matrix A has block Toeplitz-plus-Hankel with Toeplitz-plus-Hankel blocks (BTHTHB) structure. It is shown that for the symmetric PSF, the two-dimensional discrete cosine transform (DCT-III) can be applied to diagonalize the blurring matrix A [7]. The antireflective BCs is constructed by the antireflection of FOV data to outside. In this BCs, the blurring matrix structure is block Toeplitz-plus-Hankel-plus-rank-2-correction. The discrete sine transform (DST-I) can be used to diagonalize the coefficient matrix A for the symmetric PSF [9]. The mean BCs can be viewed as an adaptive antireflection. This BCs reduces the ringing effects and keeps the $\mathcal{C}^1$ continuity. The Kronecker product approximations method has been presented to implement the image restoration process with different BCs such as whole-sample symmetric BCs, reflective BCs, antireflective BCs and mean BCs [10–13]. Since the proposed approximation does not require the symmetry condition of PSF, we apply it to implement the computations in the mean BCs [13].

In general, since the linear system (1.4) is ill-conditioned, to reduce the number of iterations when using the precondition technique with iteration methods, the standard approach to preconditioning cannot be used. One approach for preconditioning such ill-conditioned problems is to construct a matrix P that clusters the large singular values around one, but leaves the small singular values alone, one can see [14] for more details. In this paper, the Tikhonov regularization method [14–16] is used to solve this linear system, by transforming it into the following equivalent problem:

$$\min_f ||Af - g||_2^2 + \mu^2||Lf||_2^2,$$

where $0 < \mu < 1$ is a penalty parameter and L is an auxiliary operator and chosen as the identity matrix. Other effective techniques for solving image restoration problems, one may refer to [17–20]. To attain its minimum, we turn to solve the following normal equation

$$(A^TA + \mu^2 I)f = A^Tg, \tag{1.5}$$

which is equivalent to the $2n^2$-by-$2n^2$ linear system

$$\underbrace{\begin{bmatrix} I & A \\ -A^T & \mu^2 I \end{bmatrix}}_{\bar{\mathcal{K}}} \underbrace{\begin{bmatrix} e \\ f \end{bmatrix}}_{\bar{x}} = \underbrace{\begin{bmatrix} g \\ 0 \end{bmatrix}}_{\bar{b}}, \tag{1.6}$$

where $e = g - Af$. It is worth noting that the matrix A arising from image restoration problem (1.4) is highly structured and severely ill-conditioned, having the property that the singular values decay to, and cluster at zero. Thus, to accelerate the rate of convergence with the case $L \neq I$, we can use two BCCB matrices P_A and P_L which are the approximation of matrices A and L, respectively, to precondition the corresponding normal system $(A^TA + \mu^2 L^TL)f = A^Tg$, where $P_A = F^*\Lambda_A F$ and $P_L = F^*\Lambda_L F$ denote the FFT of corresponding matrices and the elements of diagonal matrices Λ_A and Λ_L are made up of eigenvalues of P_A and P_L, respectively. Additionally, by recasting equivalently the original system (1.4), employing the Tikhonov regularization method, into the $2n^2$-by-$2n^2$ linear system (1.6), the magnitude of singular values (i.e., the behaviour of ill-conditioned) of latter system can be greatly improved. Then the precondition technique is therefore considered in the present paper. The detailed theoretical analysis concerning the singular values of coefficient matrix $\bar{\mathcal{K}}$ is given in the Appendix part. The linear system (1.6) can be regarded as a special case of augmented systems, for which many efficient iteration methods have been presented in the literature. Examples of such methods include Uzawa-type methods [21–29], Hermitian and skew-Hermitian splitting (HSS) methods [30–34], matrix splitting iterative methods [35–37], relaxation iterative methods [38], restrictive preconditioners for conjugate gradient (RPCG) methods [39,40], Krylov subspace iterative methods combined with block-diagonal, block-tridiagonal, constraint, SOR and HSS preconditioners [41–44], iterative null space methods [45,46] and the references therein.

Recently, based on the HSS iteration method [48], Lv et al. [49] established a special HSS (SHSS) iteration method, which is different from the HSS iteration method, for solving the linear system in image restoration problem (1.6). Subsequently, Aghazadeh et al. [50] split the Hermitian part $H = \frac{1}{2}(A + A^T)$ of A as sum of two matrices G and P, where G is