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Strict upper and lower bounds for quantities of interest in static response sensitivity analysis

Mengwu Guo*, Hongzhi Zhong

Department of Civil Engineering, Tsinghua University, Beijing 100084, China

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ABSTRACT

In this paper, a goal-oriented error estimation technique for static response sensitivity analysis is proposed based on the constitutive relation error (CRE) estimation for finite element analysis (FEA). Strict upper and lower bounds of various quantities of interest (QoI) that are associated with the response sensitivity derivative fields are acquired. Numerical results are presented to assess the strict bounding properties of the proposed technique.

1. Introduction

In the design of engineering structures, the finite element method (FEM) has been widely used to make critical decisions. In order to control the quality of numerical simulations and develop confidence in decisions, a research topic, referred to as *model verification*, has been intensively studied for more than four decades. Among different error sources of numerical simulations for a chosen model, the discretization error is predominant and controllable. For the purpose of evaluating discretization error in finite element analysis (FEA), several families of *a posteriori* error estimators [1–4] have been presented for the estimation of errors measured in global norms, such as explicit error estimators [5], implicit error estimators [6,7], recovery-based error estimators [8], hierarchical estimators [9], *constitutive relation error* (CRE) estimators [10], etc.

The goal of many finite element computations in structural analysis is the determination of some specific quantities of interest, such as local stress values, displacements etc., which is necessary for a particular design decision. Thus, it is frequently the case that *a posteriori* finite element error analysis is focused on goal-oriented error estimation. Towards this end, adjoint/dual-based techniques are used to estimate the errors in solution outputs, which have been systematically reviewed in [11–14]. Research on goal-oriented error estimation was initiated in the 1990s [15–22]. Since then, several methods have been developed and applied to solutions of various problems, such as Poisson's equation, linear and non-linear static problems in solid mechanics, eigenvalue problems, time-dependent problems, non-trivial problems of CFD, etc (see [23, 24] for example). A variety of specific error estimation techniques have been proposed to evaluate the discretization error in quantities of interest, for instance, the adjoint-weighted residual method [11,14,23], the energy norm based estimates [25], the Green's function decomposition method [26], the strict-bounding approach based on Lagrangian formulation [27], the CRE-based error estimation [20]. Among the available techniques, the CRE-based error estimation provides guaranteed

* Corresponding author.
E-mail addresses: gmw13@mails.tsinghua.edu.cn (M. Guo), hzz@tsinghua.edu.cn (H. Zhong).

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strict bounds of quantities. The strict bounding property, together with its advantage of wide applicability [28–38], makes the CRE stand out for goal-oriented error estimation.

Sensitivity analysis plays an important role in uncertainty analysis, structural optimization and many other areas of structural analysis. When using some numerical methods in the first-order perturbed formulation to compute the static response sensitivity of a structural system, discretization error exists in the analysis. For instance, the stochastic perturbation method [39] is usually chosen to obtain statistically characteristic values of some structural outputs, in which the response sensitivity derivatives with respect to input parameters appear in the expressions of coefficients. Hence controlling the discretization error in response sensitivity helps enhance the accuracy of evaluating the statistically characteristic outputs. In the context of structural optimization or other parameterized problems that require repeatedly solving the structural responses under different inputs, gradient-based algorithms desire the response sensitivity derivatives at each iteration step in the parameter space. If some reduced order methods, such as the reduced basis method [40] and the proper generalized decomposition [41], are used to solve the structural responses and response derivatives at a number of sampling points with a decreased computational cost, the verification of numerical simulations will also play a crucial part throughout the procedure, see [40,42,43] for examples. Therefore, *a posteriori* estimators are required to estimate the discretization error in the solution for sensitivity derivatives of the structural response, especially in some specific quantities about the response sensitivity. As far as the authors know, the relevant error estimation techniques have not been adequately studied, and only limited information has been available. For example, an explicit (residual-based) error estimator has been used in *a posteriori* error estimation in sensitivity analysis [44], and a Neumann-subproblem *a posteriori* finite element procedure has been proposed to provide upper and lower bounds for functionals of the response sensitivity derivative fields [45].

On the basis of the principle of minimum complementary energy, the CRE-based goal-oriented error estimation will be extended to the cases of non-symmetric bilinear forms, especially to the static response sensitivity analysis of linear structural systems by the FEM in this paper. Consequently, strict upper and lower bounds can be obtained for quantities of interest, which are linear functionals associated with the sensitivity derivative fields of displacements, including the sensitivity derivatives of some scalar-valued static response quantities.

Following the introduction, the basics of the CRE estimation and the CRE-based goal-oriented error estimation are reviewed in Section 2. In Section 3, the CRE-based error estimator is extended to the cases with non-symmetric bilinear forms, and in Section 4, the estimator is used for goal-oriented error estimation of static response sensitivity. Numerical results for some model problems are presented to assess bounding property of the proposed estimation technique in Section 5. In Section 6, conclusions are drawn.

2. Basics of the constitutive relation error estimation

2.1. An abstract primal problem

To start with, a typical problem in structural analysis is introduced [46]. A Banach space $\mathcal{V}$, referred to as the ‘space of kinematically admissible solutions’, consists of all the possible displacements that satisfy the Dirichlet boundary conditions[1]. As its dual space, the ‘loading space’ $\mathcal{V}^*$ is given with the duality pair ${}_{\mathcal{V}^*}\langle\cdot,\cdot\rangle_{\mathcal{V}}$. Usually, a load $f\in\mathcal{V}^*$ includes a body force in the domain that the structure occupies and a traction on its Neumann boundary. A Banach space of strains, $\mathcal{E}$, and its dual space – the space of stresses, $\mathcal{E}^*$, are introduced, and their duality pair is written as ${}_{\mathcal{E}^*}\langle\cdot,\cdot\rangle_{\mathcal{E}}$.

The relation between a displacement element $v\in\mathcal{V}$ and its corresponding strain $\varepsilon\in\mathcal{E}$ is represented by a linear differential operator $A:\mathcal{V}\to\mathcal{E}, v\mapsto\varepsilon$, i.e. $\varepsilon=Av$. The adjoint operator of A, denoted by $A^*:\mathcal{E}^*\to\mathcal{V}^*$, is then defined as

$$ {}_{\mathcal{E}^*}\langle\tau,Av\rangle_{\mathcal{E}} = {}_{\mathcal{V}^*}\langle A^*\tau,v\rangle_{\mathcal{V}} \quad \forall(\tau,v)\in\mathcal{E}^*\times\mathcal{V}\,. \tag{1}$$

In structural analysis, the operator A is usually gradient-like and A^* is divergence-like, which is a natural derivation from Green’s formula. Besides, the relation between stresses and strains, or termed the constitutive relation, is represented by a material operator $K:\mathcal{E}\to\mathcal{E}^*$.

Then the governing equations for the primal structural problem are given as follows:

$$ \begin{aligned} &u\in\mathcal{V}\,,\ \varepsilon\in\mathcal{E}\,,\ \sigma\in\mathcal{E}^*\,,\\ &\varepsilon=Au\,,\ \sigma=K\varepsilon\,,\ A^*\sigma=f\,, \end{aligned} \tag{2}$$

or written with a single unknown u as

$$ u\in\mathcal{V}\,,\ A^*(K(Au))=f\,. \tag{3}$$

With the aid of Eq. (1), the weak form of Eq. (3) is stated as: find $u\in\mathcal{V}$ such that

$$ {}_{\mathcal{E}^*}\langle K(Au),Av\rangle_{\mathcal{E}} = {}_{\mathcal{V}^*}\langle f,v\rangle_{\mathcal{V}} \quad \forall v\in\mathcal{V}\,, \tag{4}$$

which is also referred to as the virtual work principle.

[1] In this paper, only the problems with homogeneous Dirichlet boundary conditions are discussed, since those with nonhomogeneous Dirichlet boundary conditions can be equivalently transformed to homogeneous cases.