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# Nonlinear dynamic analysis of viscoelastic beams using a fractional rheological model

Olga Martin

*Applied Sciences Faculty, Polytechnic University of Bucharest, Romania*

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ABSTRACT

The paper presents quasi-static analysis, classical and fractional dynamic analysis of a simply supported viscoelastic beam subjected to uniformly distributed load, where the Riemann–Liouville fractional derivative is of the order $\nu \in (0, 1)$. A comparative study of the results obtained for a classical and fractional Zener model using the techniques of Laplace transform, Bessel functions theory and binomial series is achieved. The graphic representations show how the existence of fractional derivative in the selected rheological model influences the dynamic response of the structure. This paper provides a theoretical basis for researchers who want to choose a mathematical model that will precisely fit with a particular experimental model.



## 1. Introduction

In recent engineering design are frequently present the structures with viscoelastic components due to their abilities to dampen out the vibrations. A proper choice of the material constitutive model plays a fundamental role in the identification of damped structural systems. The application of fractional calculus in the study of viscoelastic materials is due to Gemant [1] and Scott Blair [2]. An extensive review of fractional differential equations is given in Podlybny [3] and a survey of applications of fractional calculus in physics including viscoelasticity is presented by Hilfer [4].

In this paper, the governing equation for a simply supported viscoelastic beam under a uniform distributed load is presented using Euler–Bernoulli theory, [5–18]. This equation is accompanied by a constitutive law defined in a hereditary integral form [19–21]. In order to obtain the quasi-static exact solution (i.e. the solution ignoring inertia effects) will be used the correspondence principle for classical Zener model [19,21]. This principle relates mathematically the solution of a linear, viscoelastic boundary value problem to an analogous problem of an elastic body of the same geometry and under the same initial boundary conditions. Mention that not all problems can be solved by this principle, but only those for which the boundary conditions do not vary with the time.

Then, using the fractional Zener model, [22–27], which is obtained by generalizing the results of the classical Zener model, the dynamic analysis of a linear viscoelastic beam is achieved. Also, the plots of the relaxation modulus in the time domain and then, written in a non-dimensional form for different values of the order of fractional derivative are provided. The governing equation is solved with a mixed algorithm based on Galerkin's method for the spatial domain and Laplace transform, Bessel functions theory and binomial series expansion for the time domain.

*E-mail address:* omartin_ro@yahoo.com

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Numerical results obtained for both classical and fractional models are accompanied by graphic representations of the solutions for the governing equation. It is also discussed the role of fractional derivative order in modifying the properties of classical models.

## 2. Problem formulation

Consider a simply supported beam of length $l$ and the cross section $S_t$ of the area $S$, subjected to uniformly distributed load $\bar{p}$. In Euler's theory [17], the geometrical equation is given by

$$\varepsilon(x,t) = z\,\frac{\partial^2 w(x,t)}{\partial x^2}, \tag{1}$$

where $x$ and $z$ represent the axial and transverse coordinates, $t$ is the time, $w$-the deflection and $\varepsilon$-the strain.

A constitutive law in the hereditary integral form, [27,21] is proposed

$$\sigma(x,t) = G(0)\varepsilon(x,t) - \int_0^t \frac{dG(t-\tau)}{d\tau}\,\varepsilon(x,\tau)\,d\tau, \tag{2}$$

where $G(t)$ is the relaxation modulus for the beam material and $\sigma$ the stress corresponding to the strain $\varepsilon$.

Accordance with the d'Alembert's principle, the governing equation is of the following form:

$$\frac{\partial^2 M(x,t)}{\partial x^2} = \bar{p} - \rho S\frac{\partial^2 w(x,t)}{\partial t^2}, \tag{3}$$

where $\rho$ is the density of the material and the bending moment $M$ is written as

$$M(x,t) = z\iint_{S_t} \sigma(x,t)\,z\,dydz. \tag{4}$$

Using (2), the bending moment in a beam of constant cross section $S_t$ is equal to

$$M(x,t) = G(0)I\frac{\partial^2 w(x,t)}{\partial x^2} - I\int_0^t \frac{dG(t-\tau)}{d\tau}\,\frac{\partial^2 w(x,t)}{\partial x^2}\,d\tau, \tag{5}$$

where $I$ is the area moment of inertia given by $\iint_{S_t} z^2\,dydz$.

Thus, after the substitution of (5) in (3), the following motion equation in terms of the transverse deflection is obtained [27]

$$\rho S\,\frac{\partial^2 w(x,t)}{\partial t^2} + G(0)I\frac{\partial^4 w(x,t)}{\partial x^4} - I\int_0^t \frac{dG(t-\tau)}{d\tau}\frac{\partial^4 w(x,\tau)}{\partial x^4}d\tau = \bar{p}. \tag{6}$$

The solving of the Eq. (6) will lead to the finding of $w$ for initial and boundary conditions corresponding to a simply supported beam.

Choosing as rheological model a classical Zener model, which consists of Hooke element in serial connection with a Kelvin–Voigt element, the viscoelastic behavior of the beam material is described by the following differential equation:

$$\sigma(t) + \frac{\eta}{k_1+k_2}\,\dot{\sigma}(t) = \frac{k_1k_2}{k_1+k_2}\,\varepsilon(t) + \frac{k_1\eta}{k_1+k_2}\dot{\varepsilon}(t), \tag{7}$$

where $k_1$, $k_2$ are the elastic modulus of the springs and $\eta$ is the coefficient of viscosity of the dashpot, [19].

Now consider that the material of the beam is in its relaxation phase. So, under the constant strain $\varepsilon=\varepsilon_0$, the stress will decrease and the solution of differential equation(7) for the condition: $\sigma(0)=k_1\,\varepsilon_0$, is

$$\sigma(t) = \frac{k_1k_2}{k_1+k_2}\left(1+\frac{k_1}{k_2}\,e^{-\frac{t}{\tau_a}}\right)\varepsilon_0 = G(t)\varepsilon_0, \tag{8}$$

where,

$$\tau_a = \frac{\eta}{k_1+k_2} \tag{9}$$

is the relaxation time. Therefore, the relaxation modulus has the form

$$G(t) = \frac{k_1k_2}{k_1+k_2}\left(1+\frac{k_1}{k_2}e^{-\frac{t}{\tau_a}}\right). \tag{10}$$

Many authors mentioned in the book of Mainardi [25] have used fractional calculus as an empirical method of describing the properties of a linear viscoelastic material. The generalization of the classical viscoelastic model (7) leads to the constitutive equation

$$\sigma(t) + \frac{\eta}{k_1+k_2}\,\frac{d^\nu\sigma(t)}{dt^\nu} = \frac{k_1k_2}{k_1+k_2}\,\varepsilon(t) + \frac{k_1\eta}{k_1+k_2}\,\frac{d^\nu\varepsilon(t)}{dt^\nu}, \tag{11}$$