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Abstract. The optimal convergence rates on fast multipole methods (FMMs) for different Green's functions are considered. Simpler convergence analysis and the more accurate convergence rates are presented.

Keywords. convergence rate, fast multipole method, multipole expansion.

1 Introduction

The fast multipole method (FMM), introduced by Rokhlin and Greengard, was developed for rapid solution of boundary integral formulations of classical potential theory (Laplace's equation) and Helmholtz equation. It speeds up the calculation of long-ranged forces in the N-body problems [3, 4, 5, 9, 10] based on the multipole expansions of the system Green's functions and treating the interactions between far-away basis functions in a hierarchical manner. As a result, the complexity of matrix-vector products in an iterative solver is improved from $\mathcal{O}(N^2)$ to $\mathcal{O}(N)$ for any given accuracy ϵ.

The FMM has expanded the area of applicability to far greater problems than were previously possible and has been regarded as one of the top ten algorithms in scientific computing that were developed in the 20th century [2].

One of the distinct advantages of the fast multipole algorithm is that it comes equipped with rigorous error estimates. For the N-body problem or particle simulations, the convergence rates have been extensively studied in [3, 4, 5, 6, 9] for Green's function $G(z_0, z) = -\log(z_0 - z)$ with errors caused by $(p+1)$-term multipole expansion E_M^p and local expansion E_L^p approximations as follows

$$E_M^p \le \left(\frac{A}{c-1}\right)\left(\frac{1}{c}\right)^p = O(c^{-p}), \quad E_L^p < \left(\frac{A[4e(p+c)(c+1)+c^2]}{c(c-1)}\right)\left(\frac{1}{c}\right)^{p+1} = O(pc^{-p}), \tag{1.1}$$

where A is the summation of charges of strength and c is a positive constant greater than one. In particular, c could be $\frac{4-\sqrt{2}}{\sqrt{2}} \approx 1.828$ [3, p. 15]. These estimates can be directly applied to continuous cases [7, 8].

This paper focuses on the optimal convergence rates on FMMs for rapid computation of the continuous case $\int_{\partial\Omega} G(z_0, z)q(z)\,\mathrm{d}s(z)$ with $G(z_0, z) = -\log(z_0 - z)$ or $G(z_0, z) = \frac{1}{(z_0-z)^\alpha}$ ($\alpha \in \mathbb{N}_+$) which involves in solving 2D potential problems, elastostatic problems and Helmholtz equations [7], where $q(z)$ is a density function and $\partial\Omega$ is a closed continuous Jordan plane curve. Particle simulations can be considered as a special case in discrete form. A simpler and more accurate convergence analysis is established and the accuracy is illustrated by some examples.

2 Convergence analysis

In the tree structure, two boxes at the same level are said to be well-separated if they are not adjoint, i.e., they have no common vertex. Points z_1 and z_2 are called well-separated if they are located in two

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