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Non-monotonic blow-up problems: Test problems with solutions in elementary functions, numerical integration based on non-local transformations

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A B S T R A C T

We consider blow-up problems having non-monotonic singular solutions that tend to infinity at a previously unknown point. For second-, third-, and fourth-order nonlinear ordinary differential equations, the corresponding multi-parameter test problems allowing exact solutions in elementary functions are proposed for the first time. A method of non-local transformations, that allows to numerically integrate non-monotonic blow-up problems, is described. A comparison of exact and numerical solutions showed the high efficiency of this method.

It is important to note that the method of non-local transformations can be useful for numerical integration of other problems with large solution gradients (for example, in problems with solutions of boundary-layer type).

1. Introduction

Difficulties arising in solving blow-up problems and some methods of numerical integration of such problems are described, for example, in [1–8]. We note that for non-monotonic blow-up problems, the hodograph transformation method [2] and methods based on some other transformations [7,8] are not applicable, since the inverse function to the solution is multivalued in such problems.

2. Test problems with non-monotonic blow-up solutions

Below, we present several nonlinear blow-up test problems and their non-monotonic exact solutions that are obtained by suitable nonlinear point transformations and differential substitutions from second-, third-, and fourth-order linear ODEs, considered in [9].

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Test problem 1. A three-parameter test Cauchy problem for a nonlinear second-order autonomous ODE is

$$x''_{tt} - 3xx'_t - 2\lambda x'_t + x^3 + 2\lambda x^2 + (\beta^2 + \lambda^2)x = 0; \tag{1}$$

$$x(0) = b\beta, \qquad x'_t(0) = 2b\beta\lambda + b^2\beta^2. \tag{2}$$

The exact solution of the problem (1)–(2) is defined by the formula

$$x = \frac{b[\lambda \sin(\beta t) + \beta \cos(\beta t)]}{e^{-\lambda t} - b \sin(\beta t)}. \tag{3}$$

This solution can change the sign and, for certain values of the parameters, is a non-monotonic blow-up solution. For example, for $b = 0.5$, $\beta = 5$, and $\lambda = 0.1$, the solution (3) has a pronounced non-monotonic character with twelve local extrema and blow-up point $t_* = 7.7730738$ (see also Section 4).

Test problem 2. A four-parameter test Cauchy problem for a second-order autonomous ODE with cubic nonlinearity is

$$xx''_{tt} - 2(x'_t)^2 - 2\lambda xx'_t + (\beta^2 + \lambda^2)x^2(ax - 1) = 0; \tag{4}$$

$$x(0) = a^{-1}, \qquad x'_t(0) = a^{-2}b\beta. \tag{5}$$

The exact solution of the problem (4)–(5) has the form

$$x = \frac{1}{a - be^{\lambda t}\sin(\beta t)}. \tag{6}$$

Varying the free parameters a, b, β, and λ it is possible to obtain non-monotonic blow-up solutions with (numerous) local extrema.

Test problem 3. A four-parameter test Cauchy problem for a third-order autonomous ODE with cubic nonlinearity

$$x^2 x'''_{ttt} - 6xx'_t x''_{tt} - 2\lambda x^2 x''_{tt} + 6(x'_t)^3 + 4\lambda x(x'_t)^2 + (\beta^2 + \lambda^2)x^2 x'_t = 0; \tag{7}$$

$$x(0) = a^{-1}, \qquad x'_t(0) = a^{-2}b\beta, \qquad x''_{tt}(0) = 2a^{-3}b\beta(a\lambda + b\beta) \tag{8}$$

has the exact solution (6) (see also Section 6).

Test problem 4. A three-parameter test Cauchy problem for a third-order autonomous ODE with fourth-degree nonlinearity is

$$x'''_{ttt} + 4xx''_{tt} + 3(x'_t)^2 + 6x^2 x'_t + (\beta^2 - \lambda^2)x'_t + x^4 + (\beta^2 - \lambda^2)x^2 - \beta^2\lambda^2 = 0; \tag{9}$$

$$\begin{gathered} x(0) = \lambda + b\beta, \quad x'_t(0) = -2b\beta\lambda - b^2\beta^2, \\ x''_{tt}(0) = -b\beta^3 + 3b\beta\lambda^2 + 6b^2\beta^2\lambda + 2b^3\beta^3. \end{gathered} \tag{10}$$

The exact solution of this problem has the form

$$x = \frac{\lambda e^{\lambda t} + b\beta \cos(\beta t)}{e^{\lambda t} + b \sin(\beta t)}. \tag{11}$$

By varying the free parameters b, β, and λ, one can obtain non-monotonic blow-up solutions.

Test problems 5 *and* 6. Two test Cauchy problems for fourth-order ODEs with non-monotonic blow-up solutions can be obtained by differentiating Eqs. (7) and (9) with respect to t (because of the cumbersomeness they are not given here).