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Attractivity for fractional differential equations in Banach space☆

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In this paper, we initiate the question of attractivity of solutions for fractional differential equations in abstract space. We establish sufficient conditions for the existence of globally attractive solutions for the Cauchy problems. Our results essentially reveal the characteristics of solutions for fractional differential equations with the Riemann–Liouville derivative.

1. Introduction

Fractional differential equations have gained considerable importance due to their widespread applications in a variety of fields such as physics, mechanics, chemistry, engineering. In recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives. For details and examples, we refer the reader to the monographs by Kilbas et al. [1], Diethelm [2], Zhou [3], and a series of papers [4–9] and the references cited therein. Recently, Chen et al. [7], Losada et al. [8] and Banaś and O'Regan [9] investigated the attractivity of solutions for fractional ordinary differential equations and integral equations. However, to the best of our knowledge, the work on the attractivity of solutions for fractional differential equations in abstract space is yet to be initiated.

In this paper, we address the question of the attractivity of solutions for a Cauchy problem of the Riemann–Liouville type fractional differential equations given by

$$\begin{cases} ({}^{L}D^{\alpha}_{0+}x)(t) = f(t, x(t)), \quad t \in (0, \infty), \\ (I^{1-\alpha}_{0+}x)(0) = x_0, \end{cases} \tag{1.1}$$

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where ${}^L D_{0+}^{\alpha}$ is the Riemann–Liouville fractional derivative of order $0 < \alpha < 1$, $I_{0+}^{1-\alpha}$ is the Riemann–Liouville fractional integral of order $1-\alpha$, $f : [0,\infty) \times X \to X$ is a continuous function satisfying some assumptions and x_0 is an element of the Banach space X.

In this paper, we study the question of the attractivity of solutions for the Cauchy problem (1.1). We establish sufficient conditions for the global attractivity for solutions of (1.1). These results essentially reveal the characteristics of solutions for fractional differential equations with the Riemann–Liouville derivative. More precisely, integer order differential equations do not have such attractivity.

2. Preliminaries

In this section, we firstly recall some concepts on fractional integrals and derivatives, and then give some lemmas which are useful in next sections.

Let $\alpha \in (0,1)$ and $u \in L^1([0,\infty), X)$. The Riemann–Liouville fractional integral is defined by

$$(I_{0+}^{\alpha} u)(t) = g_{\alpha}(t) * u(t) = \int_0^t g_{\alpha}(t-s)u(s)ds, \ t > 0,$$

where $*$ denotes the convolution and

$$g_{\alpha}(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}.$$

For $u \in C([0,\infty), X)$, the Riemann–Liouville fractional derivative is defined by

$$({}^L D_{0+}^{\alpha} u)(t) = \frac{d}{dt}(g_{1-\alpha}(t) * u(t)).$$

Proposition 2.1 ([1]). *If $a > 0$ and $b > 0$, then*

$$\int_0^t (t-s)^{a-1}s^{b-1}ds = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}t^{a+b-1}.$$

Lemma 2.1 ([3]). *Assume that the operator $f : [0,\infty) \times X \to X$ is continuous. The Cauchy problem (1.1) is equivalent to the integral equation*

$$x(t) = t^{\alpha-1}x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,x(s))ds, \quad t > 0, \tag{2.1}$$

provided the right side is point-wise defined on $(0,\infty)$.

Let $C_0([t_0,\infty), X) = \{x \in C([t_0,\infty), X) : \lim_{t\to\infty}|x(t)| = 0\}$. It is obvious that $C_0([0,\infty), X)$ is a Banach space.

We need also the following generalized Ascoli–Arzela theorem [10].

Lemma 2.2. *The set $H \subset C_0([0,\infty), X)$ is relatively compact if and only if the following conditions hold:*

(i) *for any $T > 0$, the functions in H are equicontinuous on $[0,T]$;*
(ii) *for any $t \in [0,\infty)$, $H(t) = \{x(t) : x \in H\}$ is relatively compact in X;*
(iii) $\lim_{t\to\infty}|x(t)| = 0$ *uniformly for $x \in H$.*