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Short communication

Band structure analysis of leaky Bloch waves in 2D phononic crystal plates

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ABSTRACT

A hybrid Finite Element-Plane Wave Expansion method is presented for the band structure analysis of phononic crystal plates with two dimensional lattice that are in contact with acoustic half-spaces. The method enables the computation of both real (propagative) and imaginary (attenuation) components of the Bloch wavenumber at any given frequency.

Three numerical applications are presented: a benchmark dispersion analysis for an oil-loaded Titanium isotropic plate, the band structure analysis of a water-loaded Tungsten slab with square cylindrical cavities and a phononic crystal plate composed of Aurum cylinders embedded in an epoxy matrix.

1. Introduction

Over the past two decades, propagation of elastic waves in phononic crystal (PC) plates has attracted much attention due to their unique dynamic properties such as negative refraction [1], stop-band filtering [2], cloaking [3], among the other unconventional properties. PC plates are generally made of periodically distributed inclusions in a hosting material (matrix) and, depending on the physical nature of the components, can be classified in solid-solid, fluid-fluid and mixed solid-fluid composite systems [4].

In normal ambient conditions, the atmosphere surrounding the PC plate does not induce significant radiation of energy due to the high impedance mismatch at the solid-fluid interface. Therefore, such systems are treated as being in vacuum. However, PC plates surrounded by heavier fluids require mathematical models with appropriate radiation boundary conditions. Although a number of theoretical and experimental studies can be found in literature for the band-structure analysis of PC plates immersed in vacuum [4,5], their fluid-loaded counterparts seem to have received minor attention. Early studies in this sense are represented by the works of Mace [6], Eatwell [7] and Mead [8], who investigated the radiation properties of fluid-loaded plates stiffened along one principal direction. More recently, different formulations have been proposed in which half-spaces have been modeled by means of analytical methods [9] as well as Finite Element (FEM)-based absorbing regions [10–12] and Perfectly Matched Layers [13,14], while the so-called Plane Wave Expansion (PWE) method has been used in [15–17].

The main goal behind the present paper is to develop a coupled FEM-PWE method which enables the computation of the complex wavenumber-frequency band diagram for elastic PC plates with inclusions of arbitrary shape that are in contact with perfect fluids. The proposed formulation has the major advantage of avoiding spurious modes typical of numerical methods based on a finite discretization of the semi-infinite medium. Moreover, it can be extended to the case of lossy materials. However, the method cannot be applied to the case of a PC plate with a unit cell involving different material types such as fluid and solids. In order to benchmark the method, a fluid-loaded homogeneous isotropic plate is first examined, while the band structures for the real and imaginary components of the Bloch wavenumber are shown for a 1D PC and a 2D PC plate.

2. Hybrid Finite Element-Plane Wave Expansion (FEM-PWE) method

In this section, a hybrid variational formulation is described for the PC plate of Fig. 1 with d_1- and d_2-periodicity along directions 1 and 2, respectively, thickness h and density ρ. The plate is in

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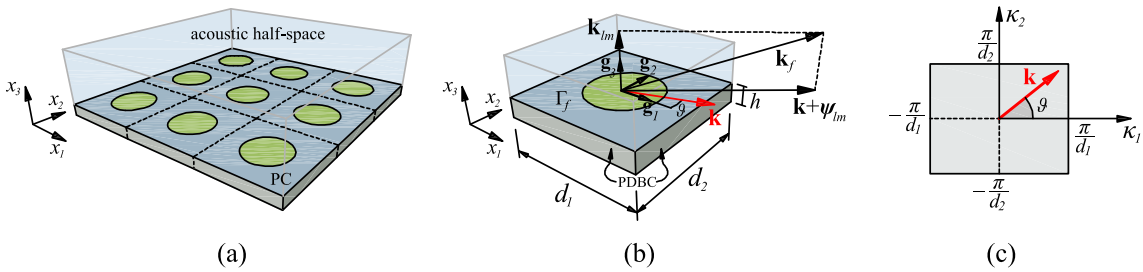

Fig. 1. PC plate model with acoustic half-space (a), primitive cell (b) and corresponding 2D reciprocal lattice (c).

contact along the surface Γ_f with a perfect fluid of density ρ_f and sound speed c_f that is infinitely extended along the direction 3. The domain of the primitive solid cell is denoted by $V = \Omega h$, where $\Omega = |d_1\mathbf{g}_1 \times d_2\mathbf{g}_2|$ while $\mathbf{g}_j$ represents the unit vector in the j-th direction.

By means of Bloch's theorem, the displacement in the solid PC and pressure in the fluid are given by $\mathbf{u}(\mathbf{x},t) = \tilde{\mathbf{u}}(\mathbf{x},t)\exp[\mathrm{i}(\mathbf{k}^{\mathrm{T}}\mathbf{x} - \omega t)]$ and $p(\mathbf{x},t) = \tilde{p}(\mathbf{x},t)\exp[\mathrm{i}\,(\mathbf{k}^{\mathrm{T}}\mathbf{x} - \omega t)]$, respectively, where i is the imaginary unit, t is time, ω denotes the angular frequency, $\mathbf{x} = [x_1, x_2, x_3]^{\mathrm{T}}$ is the configuration vector, $\tilde{\mathbf{u}}(\mathbf{x},t) = [\tilde{u}_1, \tilde{u}_2, \tilde{u}_3]^{\mathrm{T}}$ and $\tilde{p}(\mathbf{x},t)$ are Ω-periodic functions while $\mathbf{k} = [\kappa_1, \kappa_2, 0]^{\mathrm{T}} = \kappa\mathbf{\Phi}(\vartheta)$ is the Bloch wavevector, being $\mathbf{\Phi}(\vartheta) = [\cos\vartheta, \sin\vartheta, 0]^{\mathrm{T}}$ and ϑ its orientation angle with respect to the axis x_1.

Following a procedure similar to that outlined in [5] and accounting for the virtual work on the plate from the external fluid, a variational formulation for the solid PC can be stated as (the time dependency being dropped for conciseness)

$$-\int_V \rho(\mathbf{x})\omega^2(\delta\tilde{\mathbf{u}}(\mathbf{x}))^{\mathrm{H}}\tilde{\mathbf{u}}(\mathbf{x})\mathrm{d}v + \int_V (\delta\mathbf{e}(\mathbf{x},\vartheta))^{\mathrm{H}}\mathbf{C}(\mathbf{x})\mathbf{e}(\mathbf{x},\vartheta)\mathrm{d}v - \int_{\Gamma_f}(\delta\tilde{\mathbf{u}}(\mathbf{x}))^{\mathrm{H}}\tilde{p}(\mathbf{x})\mathbf{n}_3\mathrm{d}s = 0, \tag{1}$$

where $\mathbf{C}(\mathbf{x}) = C_{ijkl}(\mathbf{x})$ denotes the fourth order elasticity tensor, $\mathbf{e}(\mathbf{x},\vartheta) = \sum_{j=1}^{3}\mathbf{L}_j[\partial/\partial x_j + \mathrm{i}\kappa\mathbf{\Phi}(\vartheta)\mathbf{g}_j^{\mathrm{T}}]\tilde{\mathbf{u}}(\mathbf{x})$ indicates the Bloch strain vector, in which $\mathbf{L}_j$ are compatibility operators defined in [18]-Eq. (8) while $\mathbf{n}_3$ $(= \pm\mathbf{g}_3)$ is the outward normal at $\mathbf{x} \in \Gamma_f$.

Eq. (1) is discretized via a standard finite element discretization scheme for the solid PC, while the Plane Wave Expansion method [19,20] is used to represent the wave field in the acoustic half-space. Accordingly, the displacement field at $\mathbf{x} \in (V \cup \Gamma_f)$ is interpolated as $\tilde{\mathbf{u}}(\mathbf{x}) = \mathbf{N}(\mathbf{x})\tilde{\mathbf{q}}(\mathbf{x})$, where $\mathbf{N}(\mathbf{x})$ is a matrix of polynomial shape functions and $\tilde{\mathbf{q}}(\mathbf{x})$ denotes the vector of nodal displacements. The pressure field is expanded as $p(\mathbf{x}) = \sum_{l,m=-\infty}^{+\infty} A_{lm}\exp[\mathrm{i}(\mathbf{k} + \boldsymbol{\psi}_{lm})^{\mathrm{T}}\mathbf{x}]$, where $A_{lm} = P_{lm}\exp(\mathrm{i}\kappa_{lm}\mathbf{g}_3^{\mathrm{T}}\mathbf{x})$ and $\boldsymbol{\psi}_{lm} = [2\pi l/d_1, 2\pi m/d_2, 0]^{\mathrm{T}}$, being P_{lm} an unknown complex wave amplitude and $\kappa_{lm} = |\mathbf{k}_{lm}| = \pm[\kappa_f^2 - (\mathbf{k} + \boldsymbol{\psi}_{lm})^{\mathrm{T}}(\mathbf{k} + \boldsymbol{\psi}_{lm})]^{1/2}$, in which $\kappa_f = |\mathbf{k}_f| = \omega/c_f$ is the fluid wavenumber. By enforcing the continuity equation $\partial p(\mathbf{x})/\partial\mathbf{n}_3|_{\Gamma_f} = -\omega^2\rho_f\mathbf{n}_3^{\mathrm{T}}\mathbf{u}(\mathbf{x})$ on the fluid–solid interface and using orthogonality, the Bloch pressure and normal displacement mode functions can be expressed respectively in the form

$$\tilde{p}(\mathbf{x}) = \sum_{l,m=-\infty}^{+\infty} A_{lm}\exp\left(\mathrm{i}\boldsymbol{\psi}_{lm}^{\mathrm{T}}\mathbf{x}\right), \tag{2}$$

$$-\omega^2\rho_f\mathbf{n}_3^{\mathrm{T}}\tilde{\mathbf{u}}(\mathbf{x}) = \sum_{l,m=-\infty}^{+\infty} \mathrm{i}\kappa_{lm}A_{lm}\exp\left(\mathrm{i}\boldsymbol{\psi}_{lm}^{\mathrm{T}}\mathbf{x}\right). \tag{3}$$

The Fourier coefficients in Eq. (3) are given by

$$A_{lm}(\omega,\vartheta) = -\frac{\mathrm{i}\rho_f\omega^2}{\kappa_{lm}(\omega,\vartheta)\Omega}\int_{\Gamma_f}\mathbf{n}_3^{\mathrm{T}}\tilde{\mathbf{u}}(\mathbf{x})\exp\left(-\mathrm{i}\boldsymbol{\psi}_{lm}^{\mathrm{T}}\mathbf{x}\right)\mathrm{d}s. \tag{4}$$

Incorporating Eq. (4) into Eq. (2), substituting the resulting expression into Eq. (1) and applying a standard finite element assembling procedure over the $(1,\ldots,e,\ldots,N_e)$ elements of the mesh results in the following system of 5equations:

$$\left\{\kappa^2\mathbf{K}_3(\vartheta) + \mathrm{i}\kappa\left[\mathbf{K}_2(\vartheta) - (\mathbf{K}_2(\vartheta))^{\mathrm{T}}\right] + \mathbf{K}_1(\vartheta) - \omega^2\left[\mathbf{M} + \frac{\mathrm{i}\rho_f}{\Omega}\sum_{l,m=-\infty}^{+\infty}\frac{\mathbf{W}_{lm}\mathbf{W}_{lm}^{\mathrm{H}}}{\kappa_{lm}(\omega,\vartheta)}\right]\right\}\tilde{\mathbf{Q}}(\omega,\vartheta) = \mathbf{0}, \tag{5}$$

where $\tilde{\mathbf{Q}}(\omega,\vartheta) = \bigcup_e\tilde{\mathbf{q}}_e(\omega,\vartheta)$is the global vector of nodal displacements, $\bigcup_e$ denotes the assembling operation and

$$\mathbf{K}_1(\vartheta) = \bigcup_e\int_{V_e}\sum_{j=1}^{3}\frac{\partial(\mathbf{N}(\mathbf{x}))^{\mathrm{T}}}{\partial x_j}\mathbf{L}_j^{\mathrm{T}}\mathbf{C}_e(\mathbf{x})\mathbf{L}_j\frac{\partial\mathbf{N}(\mathbf{x})}{\partial x_j}\,\mathrm{d}v, \tag{6}$$

$$\mathbf{K}_2(\vartheta) = \bigcup_e\int_{V_e}\sum_{j=1}^{3}(\mathbf{N}(\mathbf{x}))^{\mathrm{T}}\mathbf{g}_j(\mathbf{\Phi}(\vartheta))^{\mathrm{T}}\mathbf{L}_j^{\mathrm{T}}\mathbf{C}_e(\mathbf{x})\mathbf{L}_j\frac{\partial\mathbf{N}(\mathbf{x})}{\partial x_j}\,\mathrm{d}v, \tag{7}$$

$$\mathbf{K}_3(\vartheta) = \bigcup_e\int_{V_e}\sum_{j=1}^{3}(\mathbf{N}(\mathbf{x}))^{\mathrm{T}}\mathbf{g}_j(\mathbf{\Phi}(\vartheta))^{\mathrm{T}}\mathbf{L}_j^{\mathrm{T}}\mathbf{C}_e(\mathbf{x})\mathbf{L}_j\mathbf{\Phi}(\vartheta)\mathbf{g}_j^{\mathrm{T}}\mathbf{N}(\mathbf{x})\,\mathrm{d}v, \tag{8}$$

$$\mathbf{M} = \bigcup_e\int_{V_e}(\mathbf{N}(\mathbf{x}))^{\mathrm{T}}\rho_e(\mathbf{x})\mathbf{N}(\mathbf{x})\,\mathrm{d}v, \tag{9}$$

$$\mathbf{W}_{lm} = \bigcup_e\int_{\partial V_e\in\Gamma_f}(\mathbf{N}(\mathbf{x}))^{\mathrm{T}}\mathbf{n}_3\exp\left(\mathrm{i}\boldsymbol{\psi}_{lm}^{\mathrm{T}}\mathbf{x}\right)\mathrm{d}s. \tag{10}$$

It is remarked that, in order to obey periodicity, Eq. (5) must be subjected to appropriate periodic Dirichlet boundary conditions (PDBC) on the lateral boundaries of the cell (see [5,20] for further details).

Eq. (5) is configured as a nonlinear eigenvalue problem in the complex Bloch wavenumber $\kappa(\omega,\vartheta)$ for any fixed real positive frequency ω and assigned orientation $\vartheta \in [0, 2\pi]$, and it is solved in the present work by means of a contour integral algorithm [21]. It should be noted that, while the real components of $\mathbf{k}(\omega,\vartheta)$ are restricted to the first Brillouin zone $(-\pi/d_j \leqslant \mathrm{Re}(\kappa_j(\omega,\vartheta)) \leqslant \pi/d_j$, $j = 1,2)$, the imaginary components, describing the wave decay in space along the corresponding directions, are unbounded.

Of fundamental importance in the solution of the dispersion equation is the determination of the correct sign of κ_{lm}, which is a two-valued function of the Bloch wavenumber $\kappa(\omega,\vartheta)$. The choices of $\mathrm{sgn}(\kappa_{lm})$ with physical meaning depend on the behavior of the spatial harmonic (l,m) in the acoustic region [22] and are listed in Table 1.

3. Numerical applications

In order to validate the proposed method, a benchmark analysis is first proposed for a homogeneous Titanium plate ($\rho = 4460$ kg/m^3,