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Nonlocal symmetry and exact solutions of the (2+1)-dimensional Gardner equation

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ABSTRACT

For the (2+1)-dimensional Gardner equation, the truncated Painlevé method is developed to obtain the nonlocal residual symmetry and Bäcklund transformation. And, the symmetry group transformation can be computed from the extended system. Moreover, (2+1)-dimensional Gardner equation is proved to be consistent Riccati expansion (CRE) solvable. With the help of the Riccati equation and the CRE method, we obtain the soliton–cnoidal wave interaction solution of the equation.

1. Introduction

As we all know, nonlinear evolution equations (NLEEs) and their solutions play an important role in mathematics, physics, chemistry, biology and other processes. The application of symmetry group in dealing with differential equations can be dated back to Sophus Lie [1] and various symmetry methods have been developed to find exact solutions for different partial differential equations (PDEs). In PDEs, symmetry [2–5] and Painlevé analysis [6–10] are two kinds of effective methods for constructing exact solutions. However, because of the presence of nonlocal terms, the nonlocal symmetries cannot be determined completely in an algorithmic way. In latter study, there may exist nonlocal symmetries which are obtained by inverse recursion operators [11–13], Darboux transformation (DT) [14–16], Bäcklund transformation (BT) [17], the Möbious (conformal) invariant form [18] and so on. Recently, Lou [19,20] established the consistent Riccati expansion (CRE) method, which can be used to identify CRE solvable systems and construct various interaction solutions among different nonlinear excitations. Furthermore, Lou found that the nonlocal symmetry from the truncated Painlevé expansion is just the residue of the expansion with respect to the singular manifold which called residual symmetry [21,22].

In this paper, we focus on investigating the residual symmetry and CRE solvability of the (2+1)-dimensional Gardner [23–27] equation with constants α, β, γ, which have not yet been discovered. The equation reads

$$\begin{aligned} &u_t + u_{xxx} + 6\beta u u_x - \frac{3}{2}\alpha^2 u^2 u_x + 3\gamma^2 v_y - 3\alpha\gamma u_x v = 0, \\ &v_x = u_y. \end{aligned} \tag{1}$$

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when $u_y = 0$, Eq. (1) reduces to the well known (1+1)-dimensional Gardner equation. For $\alpha = 0$, Eq. (1) is the KP equation, while it is the modified KP equation with $\beta = 0$. Therefore, the (2+1)-dimensional Gardner equation could be regarded as a combined KP and modified KP equation.

The outline of paper is organized as follows. In Section 2, the nonlocal symmetry of the Gardner equation are obtained by the Painlevé expansion approach and a new type of finite symmetry transformation is obtained by the localization process. In Section 3, the CRE property for the Gardner equation is investigated. Based on the CRE method, the explicit interaction solution between a soliton and a cnoidal periodic wave of this equation is given. Section 4 contains a summary and discussion.

2. Nonlocal symmetry and its localization

2.1. Residual symmetry via the truncated Painlevé expansion

For the (2+1)-dimensional Gardner Eq. (1), there exists a truncated Painlevé expansion

$$\begin{aligned} u &= \frac{u_0}{\phi} + u_1, \\ v &= \frac{v_0}{\phi} + v_1, \end{aligned} \tag{2}$$

with u_0, u_1, v_0, v_1, ϕ being the functions of x, y and t. Substituting (2) into (1) and vanishing all the coefficients of the same powers of $\frac{1}{\phi}$, we obtain

$$\begin{aligned} u_0 &= \frac{2}{\alpha}\phi_x, \qquad u_1 = \frac{-K\gamma}{\alpha} + \frac{2\beta}{\alpha^2} - \frac{\phi_{xx}}{\alpha\phi_x}, \\ v_0 &= \frac{2}{\alpha}\phi_y, \qquad v_1 = \frac{1}{2}\frac{\gamma K^2}{\alpha} - \frac{K_x}{\alpha} - \frac{K\phi_{xx}}{\alpha\phi_x} + \frac{1}{3}\frac{C}{\alpha\gamma} + \frac{1}{3}\frac{S}{\alpha\gamma} + \frac{2\beta^2}{\alpha^3\gamma}, \end{aligned} \tag{3}$$

and the (2+1)-dimensional Gardner Eq. (1) is successfully rewrite as its Schwartzian form

$$3K_xK\gamma^2 + 3K_y\gamma^2 + S_x + C_x = 0, \tag{4}$$

where the notations C, K and S are defined as

$$C = \frac{\phi_t}{\phi_x}, \qquad K = \frac{\phi_y}{\phi_x}, \qquad S = \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2}\frac{\phi_{xx}^2}{\phi_x^2}, \tag{5}$$

which means that Eq. (4) is form invariant under the Möbious transformation

$$\phi \to \frac{a+b\phi}{c+d\phi}, \qquad (ad \neq bc). \tag{6}$$

Due to above Möbious invariance, a well-known fact is the variable ϕ possesses the point symmetry in the form of

$$\sigma_\phi = \kappa_0 + \kappa_1\phi + \kappa_2\phi^2, \tag{7}$$

where κ_0, κ_1 and κ_2 are arbitrary constants.

From the above standard truncated Painlevé expansion, we arrive at the BT theorem as follows.

Theorem 1 (Non-auto-BT theorem). *If the function ϕ satisfies (4), then*

$$\begin{aligned} u &= \frac{-K\gamma}{\alpha} + \frac{2\beta}{\alpha^2} - \frac{\phi_{xx}}{\alpha\phi_x}, \\ v &= \frac{1}{2}\frac{\gamma K^2}{\alpha} - \frac{K_x}{\alpha} - \frac{K\phi_{xx}}{\alpha\phi_x} + \frac{1}{3}\frac{C}{\alpha\gamma} + \frac{1}{3}\frac{S}{\alpha\gamma} + \frac{2\beta^2}{\alpha^3\gamma}. \end{aligned} \tag{8}$$

Based on the residual symmetry theorem in [21], it is clear that the residual $\{u_0, v_0\}$ is a nonlocal symmetry of (1) with the solution $\{u_1, v_1\}$. Thus, Eq. (1) has a residual symmetry given by

$$\sigma^u = \phi_x, \qquad \sigma^v = \phi_y. \tag{9}$$

It is necessary to point out that the above residual symmetry is just related to the Möbious transformation symmetry (7) by the linearized equation of nonauto-BT (8).