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# Horizon thermodynamics from Einstein's equation of state

Devin Hansen [a,b,*], David Kubizňák [a,b], Robert B. Mann [b]

[a] Perimeter Institute, 31 Caroline St. N., Waterloo, ON, N2L 2Y5, Canada
[b] Department of Physics and Astronomy, University of Waterloo, Waterloo, ON, N2L 3G1, Canada

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ABSTRACT

By regarding the Einstein equations as equation(s) of state, we demonstrate that a full cohomogeneity horizon first law can be derived in horizon thermodynamics. In this approach both the entropy and the free energy are derived concepts, while the standard (degenerate) horizon first law is recovered by a Legendre projection from the more general one we derive. These results readily generalize to higher curvature gravities where they naturally reproduce a formula for the entropy without introducing Noether charges. Our results thus establish a way of how to formulate consistent black hole thermodynamics without conserved charges.



The discovery that spacetimes with horizons can be well described by thermodynamic laws [1–3] has led to much speculation about the thermodynamic meaning of gravitational field equations [4–6]. Among these, the concept of *horizon thermodynamics* emerged from the discovery that Einstein's equations on the black hole horizon can be interpreted as a thermodynamic identity [7]. First formulated for spherically symmetric black holes in Einstein gravity, horizon thermodynamics has since been extended to higher curvature gravities [8,9], time evolving [10,11] and rotating [10,12] black hole horizons, or even general null surfaces [13]. These notions have been extended to Horava–Lifshitz gravity [14], massive gravity [15], and to cosmological horizons [16].

The key idea of horizon thermodynamics is to realize that the radial Einstein equation[1], when evaluated on the black hole horizon, assumes the suggestive form

$$P = P(V,T)\,, \tag{1}$$

or in other words an *horizon equation of state*, which comes by making an assumption that the radial component of the stress-energy tensor serves as a thermodynamic pressure, $P = T^r{}_r|_{r_+}$, the temperature is identified with the Hawking temperature, $T = T_H$, and the horizon is assigned a geometric volume $V = V(r_+)$ [17,18].

By considering a *virtual displacement* of the horizon [7], the horizon equation of state can be rewritten as a *horizon first law*

$$\delta E = T\delta S - P\delta V\,, \tag{2}$$

where $S$ stands for the horizon entropy and $E$ is identified as a quasilocal energy of the black hole. For example, in Einstein gravity $E$ turns out to be the Misner–Sharp energy [19] and the obtained horizon first law (2) is a special case of the 'unified first law' discussed by Hayward [5].

While these results are rather suggestive, there are several issues in this procedure that arise upon further inspection. First, in the original derivation, it was unclear which thermodynamic variables were derived and which needed to be independently specified. The focus was previously on the provocative relation hidden within the Einstein equations when the appropriate identifications were made. Consequently this procedure provides no direct algorithmic method to derive thermodynamic properties of a spacetime where appropriate identifications are yet unknown, and has instead been used as means of highlighting the presence of known thermodynamics in the gravitational field equations.

The second issue concerns the restriction to virtual displacements $\delta r_+$ of the horizon radius. This renders the first law (2) to be of '*cohomogeneity-one*', since both $S$ and $V$ are functions only of $r_+$. Indeed (2) could just as well be written as $\delta E = (TS' + PV')\delta r_+$, with primes denoting differentiation with respect to $r_+$. This yields an *ambiguity* between 'heat' and 'work' terms and leads to a 'vacuum interpretation' of the first law (2) [12].

Here we show that both of the above dilemmas can be avoided. The key idea is to vary the horizon equation of state (1), treating the pressure $P$ and temperature $T$ as independent thermodynamic quantities. This results in a *new horizon first law*

$$\delta G = -S\delta T + V\delta P\,, \tag{3}$$

* Corresponding author.
*E-mail addresses:* dhansen@perimeterinstitute.ca (D. Hansen), dkubiznak@perimeterinstitute.ca (D. Kubizňák), rbmann@uwaterloo.ca (R.B. Mann).

[1] In this Letter we focus on the most robust formulation of horizon thermodynamics in the presence of spherical symmetry.

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which is manifestly non-degenerate and of cohomogeneity-two. Moreover, upon specifying the volume, pressure, and temperature, the horizon entropy $S$ is now a *derived concept* and so is the Gibbs free energy $G$. The standard horizon first law (2) can be recovered a-posteriori, by applying a degenerate Legendre transformation,

$$E = G + TS - PV\,. \tag{4}$$

This new derivation implies that horizon thermodynamics has considerable utility, and provides further evidence that gravitational field equations can indeed be understood as an equation of state.

We begin our discussion by briefly reviewing the traditional cohomogeneity-one approach to horizon thermodynamics in four-dimensional Einstein gravity [7], emphasizing which quantities are assumed and which can be obtained as an output. Throughout we employ the units in which $G = c = \hbar = 1$. Consider a static spherically symmetric black hole spacetime described by the geometry

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\Omega^2\,, \tag{5}$$

with a non-degenerate horizon located at $r = r_+$, determined as the largest positive root of $f(r_+) = 0$. We begin by concentrating on the case when $f(r) = g(r)$. Assuming minimal coupling to the matter, with the stress energy tensor $T_{ab}$, the radial Einstein equation evaluated on the horizon reads

$$8\pi T^r{}_r|_{r_+} = G^r{}_r|_{r_+} = \frac{f'(r_+)}{r_+} - \frac{1-f(r_+)}{r_+^2}\,, \tag{6}$$

where primes denote differentiation with respect to $r$. Identifying

$$P = T^r{}_r|_{r_+}\,, \quad T = \frac{f'(r+)}{4\pi}\,, \tag{7}$$

as the respective pressure and temperature yields

$$P = \frac{T}{2r_+} - \frac{1}{8\pi r_+^2}\,, \tag{8}$$

which is the horizon equation of state (1). Multiplying this by $4\pi r_+^2 \delta r_+$ then gives

$$\frac{\delta r_+}{2} = T\delta S - P\delta V\,, \tag{9}$$

which is the horizon first law (2), provided we either identify any one of the three quantities

$$V = \frac{4}{3}\pi r_+^3\,, \quad S = \frac{A}{4} = \pi r_+^2\,, \quad E = \frac{r_+}{2}\,, \tag{10}$$

as the volume, entropy, and energy respectively, assuming the latter is a function only of $r_+$. Identification of the remaining quantities logically follows from (2). Regardless, the obtained first law (2) is cohomogeneity one, as its every term varies solely with $r_+$, and suffers from the ambiguity of defining independent heat and work terms. However the degree of cohomogeneity in the HFL is a consequence of the procedure chosen and not intrinsic to horizon thermodynamics itself as we shall now demonstrate.

The identification of the temperature $T$ as in (7) is via standard arguments in thermal quantum field theory; it does not require any gravitational field equations. By definition the pressure is identified with the matter stress-energy as in (7). With this information the radial Einstein equation can be rewritten as

$$P = B(r_+) + C(r_+)T\,, \tag{11}$$

where $B$ and $C$ are some known functions of $r_+$ that in general depend on the theory of gravity under consideration, as does the linearity of the equation of state in the temperature $T$. Formally varying the generalized equation of state (11), we obtain

$$V\delta P = V\left(B' + C'T\right)\delta r_+ + VC\delta T\,, \tag{12}$$

upon multiplication by the geometric volume, $V(r_+)$, assuming all other parameters are fixed. It is now straightforward to rewrite this equation as

$$V\delta P = S\delta T + \delta G\,, \tag{13}$$

where

$$\begin{aligned} G &= \int^{r_+} dx V(x) B'(x) + T\int^{r_+} dx V(x) C'(x) \\ &= PV - ST - \int^{r_+} dx V'(x) B(x)\,, \\ S &= \int^{r_+} dx V'(x) C(x)\,, \end{aligned} \tag{14}$$

using integration by parts. Since (by postulate) we have identified $T$ with temperature, $P$ with pressure, and $V$ with volume, we therefore conclude that $S$ is the *entropy* and $G$ is the *Gibbs free energy* of the black hole. Note that these are *derived* quantities from the premises (7), and the field equations that yield (11), along with the assumption that the volume does not depend on $T$.

The relation (13) for the Gibbs free energy $G = G(P, T)$ is the cohomogeneity-two horizon first law (3), where $P$ and $T$ are independent quantities. It is valid for any gravitational theory whose field equations yield a linear relation between pressure and temperature. Note that since $G$ depends on the matter content only implicitly (via $P$ and $T$) it characterizes the gravitational theory. This is the origin of recently observed 'universality' of the corresponding phase behavior [20].

We can define the *horizon enthalpy* by the associated Legendre transformation $H = H(S, P) = G + TS$, and recover

$$\delta H = T\delta S + V\delta P\,, \tag{15}$$

which is another non-degenerate horizon first law. Likewise we can employ the Euler scaling argument, e.g. [21], to obtain

$$H = 2TS - 2VP\,, \tag{16}$$

which is the accompanying (four-dimensional) Smarr relation.

We can also make the degenerate Legendre transformation (4), whose degeneracy originates in the fact that $S$ and $V$ both being functions of $r_+$ are not independent quantities, and obtain so the 'old' cohomogeneity-one horizon first law (2).

Specifying to Einstein gravity in four dimensions, it is straightforward to identify $B(r_+) = -(8\pi r_+^2)^{-1}$ and $C(r_+) = 1/(2r_+)$ from (8), yielding from (14)

$$S = \pi r_+^2\,, \quad G = \frac{r_+}{3}(1 - \pi r_+ T)\,, \tag{17}$$

using the geometric definition (10) of the volume. This Gibbs free energy was previously derived and its phase diagrams studied in [20,22]; it is understood as $G = G(P, T)$ through the equation of state $r_+ = r_+(P, T)$, (8). Performing the degenerate Legendre transformation, (4), one finds $E = \frac{r_+}{2}$, in accordance with the previous approach.

We emphasize that the derivation of (13) depends only on the generalized equation of state having the form (11). At no point was it necessary to use the specific form of the volume $V(r_+)$. Consequently this new approach to horizon thermodynamics readily extends to higher dimensions and higher-curvature gravities. Let us demonstrate this for black holes in Lovelock gravity.

Lovelock gravity [23] is a geometric higher curvature theory of gravity that can be considered as a natural generalization of