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Purification of Gaussian maximally mixed states

Kabgyun Jeong [a,b], Youngrong Lim [a,*]

[a] Center for Macroscopic Quantum Control, Department of Physics and Astronomy, Seoul National University, Seoul 08826, Republic of Korea
[b] School of Computational Sciences, Korea Institute for Advanced Study, Seoul 02455, Republic of Korea

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A B S T R A C T

We find that the purifications of several Gaussian maximally mixed states (GMMSs) correspond to some Gaussian maximally entangled states (GMESs) in the continuous-variable regime. Here, we consider a two-mode squeezed vacuum (TMSV) state as a purification of the thermal state and construct a general formalism of the Gaussian purification process. Moreover, we introduce other kind of GMESs via the process. All of our purified states of the GMMSs exhibit Gaussian profiles; thus, the states show maximal quantum entanglement in the Gaussian regime.

1. Introduction

The principle of quantum purification means that for any mixed quantum state of a system A with a given orthonormal basis, there *exists* an orthonormal basis for ancillary system B (with at least the same dimension as that of the system A) that corresponds to the orthonormal basis of the system A. These two bases are closely related by a local unitary operation on the ancillary system B. This statement is the famous Hughston–Jozsa–Wootters (HJW) theorem [1]. For example, purification of a d-dimensional maximally mixed state (MMS) is just one of the $d \times d$-dimensional maximally entangled states (MESs) up to the local unitary operations on the ancillary system. In quantum information theory, purification is a mathematical procedure for generating a pure state from a mixed state [2–4]; however, this concept differs from the purification in which a pure state is constructed from a mixed state via many copies of mixed states with the same dimensions [5–7]. For a continuous-variable (CV) system, the concept of MMS is rather vague and still not well-understood. Instead of exhaustively considering all possible CV systems, here we focus on the Gaussian CV systems that have many practical applications in quantum optics and quantum information fields [8,9]. Note that the "Gaussian state" here means a quantum state having a Gaussian profile in the phase space, i.e., its Wigner function is a Gaussian distribution. We investigate several Gaussian maximally mixed states (GMMSs) and their purified states, i.e., the Gaussian maximally entangled states (GMESs). We call this process and its underlying principle *g-purification* and *Gaussian MMS–MES correspondence*, respectively.

It is noteworthy that any Gaussian state can be decomposed by an infinite-dimensional Fock basis, and any convex combination of quantum states gives a quantum state again. By using the correspondence, we find a *new* class of GMESs such as the (known) two-mode squeezed vacuum (TMSV) state with infinite squeezing parameter which is the purification of the thermal state with infinite temperature, as well as the g-purified MESs over Brádler's CV MMSs [10] and squeezed MMSs [11] in the Gaussian regime. These are then generalized in a single statement (see below). Furthermore, this method can be a powerful tool for Gaussian quantum information [12,13] (and references therein).

While the amount of entanglement of a given Gaussian state with a given purity (or mixedness) can be calculated [14], a GMMS in the CV regime that gives the MES via the purification process is not precisely defined. Therefore, we suggest several GMMS candidates (depicted in Fig. 1) and investigate their g-purifications explicitly. Note that the exact MMS is present only in a bounded Hilbert space. Even if we are dealing with an unbounded Hilbert space, however, we first perform the calculations in the bounded region and then take a limit of that region to infinity. Moreover, we describe an equivalence relation for GMMSs in the limit of the spectrum of the number operator $\hat{n}$ tending to infinity. Prior to the study of Gaussian MMSs or MESs, we briefly review the MMS–MES correspondence in the discrete-variable regime. The MMSs and MESs are main ingredients for the proof of existence of the additivity counterexample for the classical capacity of quantum channels [15–17]. Therefore, although there have been no practical suggestions for the proof to date, we can expect that Gaus-

* Corresponding author.
E-mail address: sshaep@gmail.com (Y. Lim).

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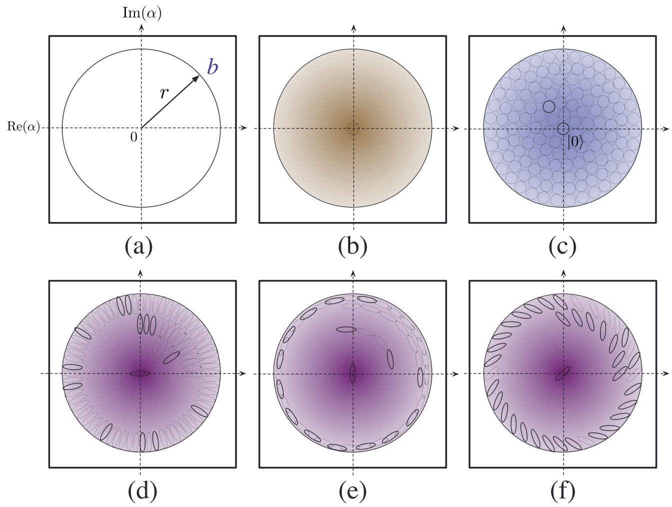

Fig. 1. Several Gaussian maximally mixed state (GMMS) candidates in the phase space within a circle boundary b. Here all candidates are depicted in $2d$ phase space, whose axes are $\mathrm{Re}(\alpha)$ and $\mathrm{Im}(\alpha)$. The radial component from the origin is expressed as r. Different colors represent the different kind of states and the density of color corresponds to the density of distribution function of state in the phase space. (a) Ideal GMMS (uniform distribution), (b) thermal state with a given temperature, (c) Brádler's continuous-variable MMS. Small circles are displaced coherent states, (d) squeezed GMMS with argument $\phi = 0$, (e) $\phi = \frac{\pi}{2}$ and (f) $\phi = \frac{\pi}{4}$. Various shapes of squeezed circles are illustrated as direction of squeezing. These all exhibit different profiles within the boundary but become identical as the boundary tends to infinity. We note that the thermal state (b) has an infinite tail thus it tends to a uniform distribution only if the temperature approaches infinity. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

sian MMS–MES correspondence can be applied to the Gaussian channel-capacity problem. Since Gaussian states are well known and can be implemented in quantum optics, we consider this Gaussian MMS–MES correspondence as a tool for experimental proof of super-additivity of the classical channel-capacity problem. Here, we assume that a GMMS has the maximal von Neumann entropy in the same manner that a full-ranked d-dimensional MMS has the maximal entropy $\log d$.

2. Gaussian MMS–MES correspondence via g-purifications

We now briefly review the purification process in the discrete-variable case in order to set the stage for our investigation on the Gaussian CV case. Suppose that a mixed state ρ_A can be decomposed by an orthonormal basis $\{|i\rangle_A\}_{i=1}^d$ such that $\rho_A = \sum_i p_i |i\rangle\langle i|_A$. To purify ρ_A, let us introduce an ancillary system B with the orthonormal basis $\{|i\rangle_B\}_{i=1}^d$ whose dimension is same as that of the system A. If we define a pure state as

$$|\psi\rangle_{AB} := \sum_{i=1}^{d} \sqrt{p_i}\, |i\rangle_A\, |i\rangle_B\,, \tag{1}$$

then we naturally obtain the reduced density matrix of the system A as ($\psi_{AB} := |\psi\rangle\langle\psi|_{AB}$)

$$\begin{aligned}\mathrm{Tr}_B(\psi_{AB}) &= \sum_{i,j=1}^{d} \sqrt{p_i p_j} |i\rangle\langle j|_A \delta_{ij} \\ &= \sum_i p_i |i\rangle\langle i|_A = \rho_A .\end{aligned} \tag{2}$$

Thus, for some fixed basis, $|\psi\rangle_{AB}$ is a purification of ρ_A. Now, suppose that $|\Psi\rangle_{AB} = \frac{1}{\sqrt{d}} \sum_{\ell=1}^d |\ell\rangle_A \otimes |\ell\rangle_B$ is a d^2-dimensional MES, we then obtain ($\Psi_{AB} := |\Psi\rangle\langle\Psi|_{AB}$)

$$\begin{aligned}\mathrm{Tr}_B(\Psi_{AB}) &= \frac{1}{d} \sum_{\ell,m=1}^{d} |\ell\rangle\langle m|_A \delta_{\ell m} \\ &= \frac{\mathbb{1}_A}{d} := \rho^A_{d,\mathrm{MMS}},\end{aligned} \tag{3}$$

where $\mathbb{1}$ denotes the d-dimensional identity matrix. This implies that the MES $|\Psi\rangle_{AB}$ is one of the purifications of $\rho^A_{d,\mathrm{MMS}}$. The d-dimensional MMS $\rho^A_{d,\mathrm{MMS}}$ has an important property that is maximal von Neumann entropy, i.e., $S(\rho^A_{d,\mathrm{MMS}}) = -\mathrm{Tr}\left(\frac{\mathbb{1}_A}{d} \log \frac{\mathbb{1}_A}{d}\right) = \log d$, where $S(\varrho) := -\mathrm{Tr}\varrho \log \varrho$. This is crucial for quantum cryptographic protocols and the theory of quantum channel-capacity.

We now consider the Gaussian CV case. In general, a d-mode Gaussian quantum system is described in $2d$-dimensional (real) symplectic phase space $\mathrm{Sp}(2d, \mathbb{R})$ and exists in the infinite dimensional Hilbert space with continuous eigenvalues of Gaussian observables [12]. For convenience, we limit our discussion on the phase space with $d = 1$, i.e., $\mathrm{Sp}(2, \mathbb{R})$. In the Gaussian regime, the concept of GMMS is not well-defined, in other words, the state cannot be uniquely specified. Prior to the main observation, we introduce an *ideal* GMMS, denoted as ρ_{GMMS} (see Fig. 1(a)), which can be expressed by an equiprobable basis set, i.e., uniform distribution in the phase space. The distribution should also have a Gaussian profile, however, it becomes uniform only in the limiting case. All other candidate states should also tend to the uniform distribution as the boundary parameter approaches the limiting value. We must be aware that the state mentioned above is a quantum state, i.e., $\mathrm{Tr}(\rho_{\mathrm{GMMS}}) = 1$, but not an identity operator $\mathbb{1}$. For a bounded basis (parameters are not tending to infinity), $\mathbb{1}$ and MMS are identical up to a constant. However, it can be easily shown that $\mathrm{Tr}(\mathbb{1}) = \infty$ in the entire phase space because of its unbounded basis; we therefore need to consider a finite region of the phase space in which a circle of radius b centered at origin and then take the limit to infinity. Note that Fig. 1 depicts several GMMS candidates in the phase space with some boundary b from the origin.

The firstly important candidate is thermal state, that can be written in the coherent state basis such as $\rho_{\mathrm{th}}(\bar{n}) = \frac{1}{\bar{n}\pi} \int e^{-\frac{|\alpha|^2}{\bar{n}}} |\alpha\rangle\langle\alpha| d^2\alpha$, where $\bar{n}$ is the mean photon number and $|\alpha\rangle$ is a coherent state. Unlike for all other cases, in the case of the thermal state (Fig. 1(b)), the temperature (variance itself) is the regularizing parameter instead of a boundary of the phase space. Therefore, we can show that an infinite temperature (infinite variance) implies that the thermal state approaches the ideal MMS. If we introduce the Gaussian operations of displacement $\hat{D}(\alpha) = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}}$ and squeezing $\hat{S}(\zeta) = e^{\frac{1}{2}(\zeta^* \hat{a}^2 - \zeta \hat{a}^{\dagger 2})}$ (where $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and the creation operators satisfying the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$), then a coherent and a squeezed coherent state, i.e., $|\alpha\rangle = \hat{D}(\alpha)\,|0\rangle \in \mathrm{Sp}(2, \mathbb{R})$ and $|\alpha, \zeta\rangle = \hat{S}(\zeta)\hat{D}(\alpha)\,|0\rangle \in \mathrm{Sp}(2, \mathbb{R})$, form an overcomplete set such that $\frac{1}{\pi} \int d^2\alpha |\alpha\rangle\langle\alpha| = \mathbb{1}$ and $\frac{1}{\pi} \int d^2\alpha |\alpha, \zeta\rangle\langle\alpha, \zeta| = \mathbb{1}$, respectively [18].

Moreover, it is important to note that the products of regularization of the convex combination of coherent or squeezed coherent states are GMMSs: for some (normalization) constants k and k', $\frac{1}{k} \sum_{i=1}^\infty \delta^2\alpha_i |\alpha_i\rangle\langle\alpha_i| = \rho^\alpha_{\mathrm{GMMS}} \in \mathrm{Sp}(2, \mathbb{R})$ and $\frac{1}{k'} \sum_{i=1}^\infty \delta^2\alpha_i \times |\alpha_i, \zeta\rangle\langle\alpha_i, \zeta| = \rho^{(\alpha,\zeta)}_{\mathrm{GMMS}} \in \mathrm{Sp}(2, \mathbb{R})$, respectively. For convenience, we omit the index i and substitute the summation by the integral as $\delta^2\alpha_i \to 0$. Then, what we need to investigate is whether $\rho_{\mathrm{GMMS}} = \rho^\alpha_{\mathrm{GMMS}} = \rho^{(\alpha,\zeta)}_{\mathrm{GMMS}} = \rho_{\mathrm{th}} \in \mathrm{Sp}(2, \mathbb{R})$.

Our main questions are: what is the purification of the ideal GMMS ρ_{GMMS} and is it a GMES? To answer these questions, we formulate a detailed Gaussian purification process (i.e., g-purifi-