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Fractional spectral vanishing viscosity method: Application to the quasi-geostrophic equation☆

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ABSTRACT

We introduce the concept of fractional spectral vanishing viscosity (fSVV) to solve conservations laws that govern the evolution of steep fronts. We apply this method to the two-dimensional surface quasi-geostrophic (SQG) equation. The classical solutions of the inviscid SQG equation can develop finite-time singularities. By applying the fSVV method, we are able to simulate these solutions with high accuracy and long-time integration with relatively low resolution. Numerical diffusion in fSVV can be tuned by the fractional order as needed. Hence, fSVV can also be applied to integer-order conservation laws that exhibit steep solutions and evolving fronts.

1. Introduction

The general 3D quasi-geostrophic equations, first derived by J.G. Charney in the 1940s [1,2], have been very successful in describing major features of large-scale motions in the atmosphere and the oceans in the mid-latitudes [3,4]. These 3D equations can be reduced to the surface quasi-geostrophic (SQG) equation with uniform potential, modeling the potential temperature on the 2D boundaries [5,6]. This paper presents a new numerical method for the SQG equation

$$\partial_t\theta + \boldsymbol{u}\cdot\nabla\theta + \kappa(-\Delta)^{\alpha}\theta = 0,\ \ \boldsymbol{x} = (x, y) \in \Omega, \tag{1.1}$$

where $\kappa \geq 0$ and $\alpha > 0$ are parameters, $\Omega \in \mathbb{R}^2$ is a bounded periodic domain, $\theta(\boldsymbol{x}, t)$ is a scalar representing the potential temperature, and $\boldsymbol{u} = (u_1, u_2)$ is the velocity field determined from $\theta(\boldsymbol{x}, t)$ by the stream function $\psi(\boldsymbol{x}, t)$ via the auxiliary relations

$$(u_1, u_2) = (-\partial_y\psi, \partial_x\psi),\ \ (-\Delta)^{\frac{1}{2}}\psi = -\theta. \tag{1.2}$$

The fractional Laplacian $(-\Delta)^{\alpha}$ in this paper is defined as follows

$$(-\Delta)^{\alpha}\theta(\boldsymbol{x}, t) = \sum_{i=1}^{\infty}\lambda_i^{\alpha}c_i(t)\phi_i(\boldsymbol{x}), \tag{1.3}$$

where $(\lambda_i, \phi_i)_{i=1}^{\infty}$ are the eigenpairs of the standard Laplacian $-\Delta$ and θ has the expansion $\theta(\boldsymbol{x}, t) = \sum_{i=1}^{\infty} c_i(t)\phi_i(\boldsymbol{x})$. Alternatively, the fractional operator $(-\Delta)^{\alpha}$ can be defined [7] through the Fourier transform

$$\widehat{(-\Delta)^{\alpha}\theta}(\omega) = \omega^{2\alpha}\widehat{\theta}(\omega), \tag{1.4}$$

where $\widehat{\theta}$ is the Fourier transform of θ [8]. When the fractional power $\alpha = \frac{1}{2}$, the equation (1.1) derived from the more general quasi-geostrophic models [9] describes the evolution of the temperature on the 2D boundary of a rapidly rotating half-space with small Rossby and Ekman numbers. Dimensionally, the 2D SQG equation with $\alpha = \frac{1}{2}$ is the analogue of the 3D Navier–Stokes equations. A general fractional order α is considered here in order to observe the minimal power of Laplacian necessary in the analysis and thus make a comparison with the 3D Navier–Stokes equations [10,11].

The inviscid SQG Eq. (1.1) (i.e., $\kappa = 0$) is useful in modeling atmospheric phenomena such as the frontogenesis i.e., the formation of strong fronts between masses of hot and cold air [5,9]. The numerical experiments show that the solution of the SQG equation with $\kappa = 0$ or $\kappa \ll 1$ emanating from very smooth initial data appears to exhibit the most singular behavior [5,12,13]. Since the solutions will develop finite-time singularities, very high resolution is required for simulations in long time intervals [14], making such computation very expensive. In this paper, we first introduce the fractional spectral vanishing viscosity (fSVV) method for solving the SQG equation in cases of inviscid ($\kappa = 0$) and inviscid-limit ($\kappa \ll 1$). The classical spectral vanishing viscosity (SVV) appears to be effective in controlling solution monotonicity while preserving spectral accuracy. It was initially developed for the resolution of hyperbolic equations using standard Fourier spectral methods [15], and later extended to large eddy simulation (LES) [16]. The standard SVV method has also been used for high Reynolds number

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incompressible flows [17,18] and for the fractional Burgers equation [19]. Following the fractional Laplacian $(-\Delta)^\alpha$ defined in Eq. (1.3), we define a new fSVV operator $S_{\mathcal{N}}^{\beta}$ based on a similar eigenfunction expansion; the exact formula of $S_{\mathcal{N}}^{\beta}$ will be given in the next section. This operator plays an important role in stabilizing the high frequency modes of the numerical solution.

The remainder of this paper is organized as follows. In Section 2 we show how to implement the fSVV method in the framework of the spectral element approximation. We propose to use an approximate form, which can be readily implemented in existing solvers [20]. The advantage of such an approximate form is that the computational cost per time-step is roughly the same with and without fSVV stabilization. In Section 3 we present the numerical results. A brief study of the influence of the fSVV tuning parameters on the convergence and accuracy is provided. Then, we consider the inviscid and viscous SQG equation with smooth initial conditions, and investigate systematically the effectiveness of the fSVV method. Finally, we provide a short summary in Section 4.

2. Numerical method

In previous work, we have developed a numerical method for computing fractional Laplacians on complex-geometry domains [20], by considering the following Eigen Value Problem (EVP) for the Laplacian:

$$-\Delta u - \lambda u = 0, \; \boldsymbol{x} \in \Omega, \tag{2.1}$$

proper boundary conditions. (2.2)

For the problems we consider here we will employ periodic boundary conditions but in principle any Dirichlet and Neumann boundary conditions can be applied. The spectral element method (SEM) [21,22] is used for solving Eqs. (2.1) and (2.2). Then, Eqs. (2.1) and (2.2) can be written in the discretized form

$$A_{\mathcal{N}}\boldsymbol{U} - \lambda M_{\mathcal{N}}\boldsymbol{U} = 0, \tag{2.3}$$

where $\mathcal{N}$ represents the number of the degrees-of-freedom (DoF) of the linear system (2.3) for the given number of elements El and polynomial degree N in each element. $A_{\mathcal{N}}$ is the corresponding matrix of the Laplacian operator under certain boundary conditions, $M_{\mathcal{N}}$ is the mass matrix, and $\boldsymbol{U}$ is the numerical solution of u. The continuous EVP is approximated by the numerical solution of the eigenpairs $(\lambda_i, \phi_i)_{i=1}^{\mathcal{N}}$ of the matrix $K = M_{\mathcal{N}}^{-1}A_{\mathcal{N}}$, and $\lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \le \lambda_{\mathcal{N}}$.

Using the numerical eigenpairs (λ_i, ϕ_i) of the Laplace operator $-\Delta$ from the SEM solution, we can approximate the fractional Laplace operator as

$$(-\Delta)^\alpha u \approx \sum_{i=1}^{\mathcal{N}} c_i \lambda_i^\alpha \phi_i, \tag{2.4}$$

where $c_i = (\phi_i, u)_{\mathcal{N}}$, and $(\cdot,\cdot)_{\mathcal{N}}$ represents discrete inner product base on Gaussian quadrature in every element. The numerical results show that this method is converging exponentially for fractional diffusion equation with smooth solutions. However, the classical solutions of the SQG equation can develop finite-time singularities with smooth initial conditions [14]. Due to this problem, we introduce a tunable fractional spectral vanishing viscosity (fSVV) method for solving the SQG equation. The variational statement of the problem reads as $\theta_{\mathcal{N}}, \psi_{\mathcal{N}} \in V_{\mathcal{N}}(\Omega)$, $\forall v \in V_{\mathcal{N}}(\Omega)$ so that

$$\left(\frac{\partial \theta_{\mathcal{N}}}{\partial t}, v\right)_{\mathcal{N}} + (\boldsymbol{u}_{\mathcal{N}} \cdot \nabla\theta_{\mathcal{N}}, v)_{\mathcal{N}} + \kappa\left((-\Delta_{\mathcal{N}})^\alpha \theta_{\mathcal{N}}, v\right)_{\mathcal{N}} + \epsilon_{\mathcal{N}}\left(S_{\mathcal{N}}^{\beta}\theta_{\mathcal{N}}, v\right)_{\mathcal{N}} = 0, \tag{2.5}$$

$$\left(u_{\mathcal{N},1}, u_{\mathcal{N},2}\right) = \left(-\partial_y\psi_{\mathcal{N}}, \partial_x\psi_{\mathcal{N}}\right), \; (-\Delta_{\mathcal{N}})^{\frac{1}{2}}\psi_{\mathcal{N}} = -\theta_{\mathcal{N}}, \tag{2.6}$$

Table 1
Kinetic Energy and Helicity for the initial data θ_0 with $\beta = 0.45$.

t	L^2-error	$K(\theta)$	$H(\theta)$	L^2-error (fSVV)	$K(\theta)$ (fSVV)	$H(\theta)$ (fSVV)
1	9.1056e−6	14.8044	26.7181	6.2618e−6	14.8044	26.7181
5	3.8480e−5	14.8044	26.7181	3.8481e−5	14.8044	26.7181
10	7.6486e−5	14.8044	26.7181	7.6485e−5	14.8044	26.7181
15	1.1459e−4	14.8044	26.7181	1.1459e−4	14.8044	26.7181
20	1.5273e−4	14.8044	26.7181	1.5273e−4	14.8044	26.7181
100	7.6328e−4	14.8044	26.7181	7.6327e−4	14.8044	26.7181

Table 2
Numerical results for the inviscid SQG equations, $\beta = 0.45$.

t	$K(\theta)$	$H(\theta)$	$K(\theta)$ (fSVV)	$H(\theta)$ (fSVV)
1	14.804407	26.718074	14.804407	26.718074
3	14.804407	26.718074	14.804407	26.718074
5	14.804407	26.718074	14.804199	26.718054
8	14.923095	26.719398	14.775744	26.715950
10	NaN	NaN	14.751368	26.714122
15	NaN	NaN	14.532270	26.695471
20	NaN	NaN	14.388240	26.681487

Table 3
Parameters used for SQG with fSVV.

	Case I	*Case II*	*Case III*	*Case IV*	*Case V*	*Case VI*
$m_{\mathcal{N}}$	$\mathcal{N}/2$	$2\mathcal{N}/3$	$\mathcal{N}/2$	$2\mathcal{N}/3$	$\mathcal{N}/2$	$2\mathcal{N}/3$
$\epsilon_{\mathcal{N}}$	$\frac{1}{N}$	$\frac{1}{N}$	$\frac{1}{N}$	$\frac{1}{N}$	$\frac{1}{N}$	$\frac{1}{N}$
β	0.8	0.8	1.0	1.0	1.2	1.2

where $\epsilon_{\mathcal{N}} = O(\frac{1}{N})$, $V_{\mathcal{N}}(\Omega) = \text{span}\{\phi_i, i = 1, \ldots, \mathcal{N}\}$ and $\beta > 0$ is a tunable fractional order. The fractional operators $(-\Delta_{\mathcal{N}})^\alpha$ and $S_{\mathcal{N}}^{\beta}$ are defined as follows

$$(-\Delta_{\mathcal{N}})^\alpha u = \sum_{i=1}^{\mathcal{N}} \lambda_i^\alpha u_i \phi_i, \quad S_{\mathcal{N}}^{\beta} u = \sum_{i=1}^{\mathcal{N}} \tilde{\lambda}_i^{\beta} u_i \phi_i, \tag{2.7}$$

where $u_i = (u, \phi_i)_{\mathcal{N}}$ and

$$\tilde{\lambda}_i = \begin{cases} 0, & i \le m_{\mathcal{N}}, \\ \exp\left(-\left(\frac{d-i}{m_{\mathcal{N}}-i}\right)^2\right)\lambda_i, & i > m_{\mathcal{N}}. \end{cases} \tag{2.8}$$

Here $m_{\mathcal{N}}$ can have different forms $m_{\mathcal{N}} = \{\sqrt{\mathcal{N}}, \frac{\mathcal{N}}{2}, \text{or } \frac{2\mathcal{N}}{3} \text{ etc.}\}$ [16,23]. Of course, the usual spectral approximations of Eqs. (1.1) and (1.2) are recovered when $\epsilon_{\mathcal{N}} = 0$ or $m_{\mathcal{N}} = \mathcal{N}$.

Next, the ordinary differential Eqs. (2.5) and (2.6) are discretized by the second-order Crank–Nicolson scheme. Let L be the number of the time steps to integrate up to final time T, then $\Delta t = T/L$. We denote by superscripts the time levels and set the initial condition $\theta_{\mathcal{N}}^0 = \theta_{\mathcal{N}}(\boldsymbol{x}, 0)$ and $\theta_{\mathcal{N}}^{-1} = \theta_{\mathcal{N}}^0$. Here, we simulate with a first-order scheme in the first time step. We look for solution of $(\theta_{\mathcal{N}}^{n+1}, \boldsymbol{u}_{\mathcal{N}}^{n+\frac{1}{2}}, \psi_{\mathcal{N}}^{n+\frac{1}{2}})$ for $n = 0, \ldots, L-1$. We introduce the following notation for convenience:

$$\theta_{\mathcal{N}}^{n+\frac{1}{2}} = \frac{1}{2}(\theta_{\mathcal{N}}^{n+1} + \theta_{\mathcal{N}}^{n}), \quad \theta_{\mathcal{N}}^{*,n+\frac{1}{2}} = \frac{1}{2}(3\theta_{\mathcal{N}}^{n} - \theta_{\mathcal{N}}^{n-1}).$$

Then, the fully discrete scheme of the SQG equation can be written as follows

$$\left(\frac{\theta_{\mathcal{N}}^{n+1} - \theta_{\mathcal{N}}^{n}}{\Delta t}, v\right)_{\mathcal{N}} + (\boldsymbol{u}_{\mathcal{N}}^{n+\frac{1}{2}} \cdot \nabla\theta_{\mathcal{N}}^{*,n+\frac{1}{2}}, v)_{\mathcal{N}} + \kappa\left((-\Delta_{\mathcal{N}})^\alpha \theta_{\mathcal{N}}^{n+\frac{1}{2}}, v\right)_{\mathcal{N}} + \epsilon_{\mathcal{N}}\left(S_{\mathcal{N}}^{\beta}\theta_{\mathcal{N}}^{n+\frac{1}{2}}, v\right)_{\mathcal{N}} = 0, \tag{2.9}$$

$$\boldsymbol{u}_{\mathcal{N}}^{n+\frac{1}{2}} = \left(-\partial_y\psi_{\mathcal{N}}^{n+\frac{1}{2}}, \partial_x\psi_{\mathcal{N}}^{n+\frac{1}{2}}\right), \; (-\Delta_{\mathcal{N}})^{\frac{1}{2}}\psi_{\mathcal{N}}^{n+\frac{1}{2}} = -\theta_{\mathcal{N}}^{*,n+\frac{1}{2}}. \tag{2.10}$$

By the orthogonality of the eigenfunctions we obtain