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Functional equations for orbifold wreath products

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ABSTRACT

We present generating functions for extensions of multiplicative invariants of wreath symmetric products of orbifolds presented as the quotient by the locally free action of a compact, connected Lie group in terms of orbifold sector decompositions. Particularly interesting instances of these product formulas occur for the Euler and Euler–Satake characteristics, which we compute for a class of weighted projective spaces. This generalizes results known for global quotients by finite groups to all closed, effective orbifolds. We also describe a combinatorial approach to extensions of multiplicative invariants using decomposable functors that recovers the formula for the Euler–Satake characteristic of a wreath product of a global quotient orbifold.

1. Introduction

If M is a manifold, the nth *symmetric product* of M is the quotient of the Cartesian product M^n by the action of the symmetric group by permuting factors. Characteristic numbers of symmetric products of manifolds have been widely studied, and their structure naturally leads to generating functions of the infinite product and exponential type, see e.g. [1–3]. If M is in addition equipped with the action of a finite group so that M/G is a global quotient orbifold, the *wreath symmetric product* is the natural generalization of the symmetric product. In the literature one can find several approaches to characteristic numbers of wreath symmetric products, see e.g. [4–8]. Many examples of orbifolds, however, are not global quotients of a manifold by a finite group. The most well-known examples are the weighted complex projective spaces, see e.g. [9–13]. Characteristic numbers of wreath symmetric products for non-global quotient orbifolds have been studied e.g. in [14–18].

In [4] and [5], Tamanoi introduced a number of orbifold invariants for global quotient orbifolds, i.e. orbifolds given by the quotient of a manifold by a finite group, generalizing the orbifold Euler characteristics of [19] and [20]. The basic idea behind these invariants is to apply a multiplicative orbifold invariant φ, e.g. the Euler–Satake characteristic (see [18]), to a Γ-sector

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decomposition of the orbifold, yielding an extension φ_Γ of this invariant. Tamanoi introduced sector decompositions of global quotients associated to an arbitrary group Γ, a Γ-set X, and a finite covering space $\Sigma' \to \Sigma$ of a connected manifold Σ with fundamental group Γ. See also [6] for connections between these extensions and the orbifold elliptic genus, which for non global quotient orbifolds was introduced in [21].

In [22], for a finitely generated discrete group Γ, the authors introduced the Γ-sectors associated to a Lie groupoid $\mathcal{G}$, which generalized Tamanoi's Γ-sector decomposition to the case of an arbitrary orbifold. The relationship between this construction, quotient presentations of orbifolds, and generalized loop spaces for orbifolds was studied in [17]. In [18], this relationship was used to extend Tamanoi's generating functions for the extension by free abelian groups $\Gamma = \mathbb{Z}^\ell$ of the Euler and Euler–Satake characteristics of wreath symmetric product orbifolds to the case of closed effective orbifolds by using their presentation as the quotient of a closed manifold by the locally free action of a compact, connected Lie group [23, Theorem 2.19]. See also [7] for the generating function for the stringy orbifold Euler characteristic, corresponding to $\Gamma = \mathbb{Z}^2$.

In this paper, we generalize Tamanoi's generating function for the Γ-extension φ_Γ of a wreath product of a multiplicative invariant φ to the case of quotients by compact, connected Lie groups acting locally freely; see Theorem 3.3. This generalizes the generating functions given in [18] from the case where φ is the Euler or Euler–Satake characteristic and $\Gamma = \mathbb{Z}^\ell$ to general φ and an arbitrary finitely generated discrete group Γ. For specific multiplicative invariants φ, these formulas relate the values of an extension φ_Γ of φ on the wreath symmetric products $M^n \rtimes G(\mathcal{S}_n)$ to values of extensions φ_H of φ on $M \rtimes G$; see Theorems 3.5 and 3.6. These expansion formulas admit both an infinite product and an exponential form. For a geometric interpretation, note that if Σ is a manifold with fundamental groups Γ, then each factor in the infinite product corresponds to a connected covering space associated to a finite index subgroup H of Γ. This generalization requires defining sector decompositions associated to Γ-sets in this generality. We illustrate these results by calculating the Γ-Euler and Γ-Euler–Satake characteristics, where Γ is the fundamental group a closed, orientable surface of positive genus, of wreath symmetric products of an interesting class of orbifold examples that are not global quotients: weighted projective spaces with weights (m, mn, n) where m and n are relatively prime.

In the last section, emphasizing the exponential form of the wreath product expansions, we introduce for global quotient orbifolds a modification of methods of Dress and Müller [24] for decomposable functors to relate the Γ-extension of the Euler–Satake characteristic and the (Γ/H)-extensions, see Theorem 4.2. Indeed their methods provide a counting algorithm for invariants of exponential type, and using this method, we recover Theorem 3.6 for global quotients. Late in the preparation of this paper, the authors became aware of [25], from which the results of this section also follow by choosing the Euler–Satake characteristic as a weight. The method of proof is similar to that in [25].

The outline of this paper is as follows. In Section 2, we collect background material on K–G-bundles for groups K and G and review the classifications given in [5]. Wreath products appear naturally in the geometric context of K–G-bundles, which are hence a standard tool to study homomorphisms into wreath products. In Section 3, we generalize Tamanoi's generating functions for Γ-extensions of multiplicative invariants of wreath symmetric products to orbifolds presented as quotients by compact, connected Lie groups acting locally freely; see Theorem 3.3. This provides a geometric context for the previous section. We apply these results to the Euler and Euler–Satake characteristics in Section 3.2, resulting in generating functions for these invariants given in Theorems 3.5 and 3.6. We illustrate Theorems 3.5 and 3.6 in Section 3.3 for the class of non-global quotient orbifolds given by weighted projective spaces with weights (m, mn, n) with $n > 1$ and $m > 1$ relatively prime and Γ the fundamental group of a closed orientable two surface of positive genus; the standard non-global quotient 2-dimensional teardrops $\mathbb{P}(1, n)$ and $\mathbb{P}(1, m)$ appear as sectors of these spaces. We then study extensions of invariants associated to arbitrary finite Γ-sets. In Section 4, we recover Theorem 3.6 for global quotients using a formal functional equation of Dress and Müller [24] for decomposable functors.

By a *quotient orbifold*, we mean an orbifold that admits a presentation as a translation groupoid $M \rtimes G$ where M is a smooth manifold and G is a Lie group acting locally freely in such a way that $M \rtimes G$ is Morita equivalent to an orbifold groupoid, see [26]. For brevity, we refer to $M \rtimes G$ as a *cc-presentation* when in addition G is compact and connected, M is closed, and the action of G on M is effective. In particular, note that an orbifold that admits a cc-presentation is compact and does not have boundary in the orbifold sense. All manifolds, orbifolds, and group actions are assumed smooth. We use χ to denote the (usual) Euler characteristic of the orbit space and χ_{ES} to denote the Euler–Satake characteristic, see [18]. Unless stated otherwise, we will always use M to denote a smooth, closed manifold, G to denote a compact Lie group, and Γ to denote a finitely generated discrete group.

2. Classifications of K–G-bundles and conjugacy classes of homomorphisms

In this section, we review results on the classifications of K–G-bundles parametrized by conjugacy classes of homomorphisms. We assume that G is a compact Lie group and K is a topological group.

Definition 2.1 (*[5,27,28]*)**.** Let X be a topological space.

(i) A K–G-bundle over X is a locally trivial G-bundle $p : P \to X$ with left K-actions on P and X such that the projection map p is K-equivariant.