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Four-body central configurations with adjacent equal masses

Yiyang Deng [a,*], Bingyu Li [b], Shiqing Zhang [a]

[a] Department of Mathematics, Sichuan University, Chengdu 610064, China
[b] College of Management Science, Chengdu University of Technology, Chengdu 610059, China

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ABSTRACT

For any convex non-collinear central configuration of the planar Newtonian 4-body problem with adjacent equal masses $m_1 = m_2 \neq m_3 = m_4$, with equal lengths for the two diagonals, we prove it must possess a symmetry and must be an isosceles trapezoid; furthermore, which is also an isosceles trapezoid when the length between m_1 and m_4 equals the length between m_2 and m_3.

1. Introduction

It is well known [1,2] that the central configurations for Newtonian n-body problems play an important role in the celestial mechanics. Here, we only consider the planar Newtonian 4-body problem.

The planar Newtonian 4-body problem is related with the motion of 4 point particles with positive masses $m_i \in \mathbb{R}$ and position vectors $q_i \in \mathbb{R}^2$ for $i = 1, \ldots, 4$, moving according to Newton's second law and the universal gravitational law:

$$m_i \ddot{q}_i = \frac{\partial U(q)}{\partial q_i}, \quad i = 1, \ldots, 4, \tag{1}$$

where

$$U(q) = G \sum_{i<j}^{4} \frac{m_i m_j}{r_{ij}} \tag{2}$$

is the Newtonian potential for the 4-body and $r_{ij} = \|q_i - q_j\|$, in the following, we let $G = 1$.

* Corresponding author.
E-mail addresses: dengyiyang@126.com (Y. Deng), sbtgvpgf@163.com (B. Li), zhangshiqing@msn.com (S. Zhang).

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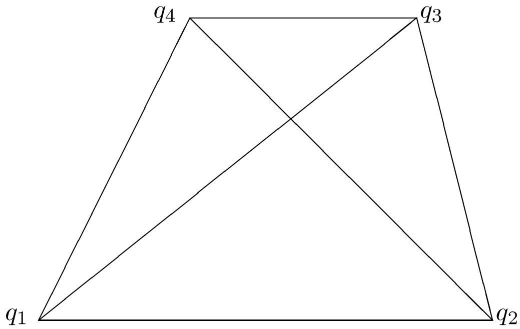

Fig. 1. Co-planar 4-body convex central configuration.

Let $q = (q_1, \ldots, q_4) \in (\mathbb{R}^2)^4$ and M be the diagonal mass matrix diag$(m_1, m_1, \ldots, m_4, m_4)$, then the system (1) can be rewritten as the following:

$$\ddot{q} = M^{-1}\frac{\partial U(q)}{\partial(q)}. \tag{3}$$

To study this problem, without lose of generality, we assume the center of mass is fixed at the origin and consider the space

$$\Omega = \left\{ q = (q_1, q_2, q_3, q_4) \in (\mathbb{R}^2)^4 \mid \sum_{i=1}^{4} m_i q_i = 0 \right\}.$$

Let $\Delta = \bigcup_{i\neq j}\{q \mid q_i = q_j\}$ be the **collision set**. The set $\Omega \setminus \Delta$ is called the **configuration space**.

Here we recall the definition of the central configuration:

Definition (*[1,2]*). A configuration $q \in \Omega \setminus \Delta$ is called a central configuration if there is some constant λ such that

$$M^{-1}\frac{\partial U}{\partial q} = \lambda q. \tag{4}$$

The recent researches of co-planar 4-body central configurations are listed in the following: In 1995 and 1996, Albouy [3,4] proved that there are exactly four equivalent classes for the central configurations of the planar Newtonian 4-body problem with positive equal masses. In 2002, Long and Sun [5] showed that any convex non-collinear central configurations of the planar 4-body problem with equal opposite masses, such that the length of the diagonal between the bigger equal opposite masses is not longer than the diagonal between the less equal opposite masses, must possess a symmetry and forms a kite. Furthermore, it must be a rhombus. In 2003, Albouy [6] showed that any convex non-collinear central configurations of the planar 4-body problem cannot be a kite when the opposite masses are not equal. In 2007, Perez-Chavela and Santoprete [7] generalized the result of Long and Sun [5] and obtained the symmetry of central configurations with equal masses located at opposite vertices of a quadrilateral, but they assume that the two equal masses are not the smallest in all masses. In 2008, Abouy, Fu and Sun [8] proved that, in the planar 4-body problem, a convex central configuration is symmetric with respect to one diagonal if and only if the masses of the two particles on the other diagonal are equal. They also showed that the less massive one is closer to the former diagonal. In this paper, they raised a question: Does the equality of two pairs of adjacent masses implies the configuration is an isosceles trapezoid for co-planar 4-body convex central configurations? (See Fig. 1.)

To solve this problem, in 2012, Cors and Roberts [9] used mutual distances as coordinates to study the four-body co-circular central configurations. They had proved that the set of positions that yield co-circular central configurations with positive masses is a two-dimensional surface, the graph of a differentiable function over two of the exterior side-lengths. The boundaries of this surface correspond to three important symmetric cases: a kite, an isosceles trapezoid and a degenerate case where three bodies lie at the vertices of an equilateral triangle and the fourth body of the quadrilateral has zero mass; furthermore, they got a stronger result, they only assume that one pair of adjacent masses is equal, then the configuration is an isosceles trapezoid for 4-body convex co-circular central configurations. In 2014, Corbera and Llibre [10] showed that there is a unique convex planar central configuration which has two pairs of equal masses located at the adjacent vertices of the configuration, and which is an isosceles trapezoid when one pair of adjacent masses is sufficiently small.

In this paper, we study the isosceles trapezoid central configurations for the four-body problems. Our main results are:

Theorem 1.1. *Let $q = (q_1, q_2, q_3, q_4) \in \Omega$ be a convex non-collinear central configuration with masses $(\beta, \beta, \alpha, \alpha)$, $\beta > \alpha > 0$. Suppose that the equal masses are at adjacent vertices. If*

$$r_{13} = r_{24},$$

then the configuration q must possess a symmetry, and then forms an isosceles trapezoid.