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Group interpretation of the spectral parameter. The case of isothermic surfaces

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ABSTRACT

It is well known that in some cases the spectral parameter has a group interpretation. We discuss in detail the case of Gauss–Codazzi equations for isothermic surfaces immersed in $\mathbb{E}^3$. The algebra of Lie point symmetries is 4-dimensional and all these symmetries are also symmetries of the Gauss–Weingarten equations (which can be considered as $\mathfrak{so}(3)$-valued non-parametric linear problem). In order to obtain a non-removable spectral parameter one has to consider $\mathfrak{so}(4,1)$-valued linear problem which has a 3-dimensional algebra of Lie point symmetries. The missing symmetry introduces a non-removable parameter. In the second part of the paper we extend these results on the case of isothermic immersions in arbitrary multidimensional Euclidean spaces. In order to simplify calculations the problem was formulated in terms of a Clifford algebra.

1. Introduction

Integrable systems of nonlinear partial differential equations often arise as compatibility conditions for an associated linear system, known as the Lax pair, with the so called spectral parameter (see, for instance, [1]). On the other hand, in the geometry nonlinear systems (Gauss–Codazzi equations) with associated linear problems (Gauss–Weingarten equations) are ubiquitous but the linear problem is usually without a parameter. Therefore it is natural to expect that one of symmetries of the nonlinear system is not a symmetry of the linear problem and can be used to produce the spectral parameter, provided that this parameter is non-removable (i.e., it cannot be removed by a gauge transformation). This idea has been suggested as a working criterion of integrability [2,3], soon formulated in a purely algebraic way [4,5]. Around the same time the same idea has been formulated in terms of coverings [6]. Since then many authors used Lie symmetries to obtain or discuss Lax pairs with non-removable parameter, see [7–11]. More powerful approach by using coverings has been developed in this context as well, see [12,13].

The group interpretation of the spectral parameter is rather natural but still surprisingly small number of cases is checked and confirmed. In the first part of this paper we discuss isothermic immersions in $\mathbb{E}^3$ [14–16]. The corresponding Gauss–Codazzi equations have 4-dimensional algebra of Lie point symmetries. All these symmetries are symmetries of the $\mathfrak{so}(3)$-valued linear problem (classical Gauss–Weingarten equations), as well. This is consistent with non-existence of an

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$\mathfrak{so}(3)$-valued Lax pair for isothermic surfaces. In order to obtain the classical Lax pair for isothermic surfaces one has to consider a larger matrix algebra, namely $\mathfrak{so}(4,1)$. Then, one of the symmetries of Gauss–Codazzi equations is not a symmetry of the $\mathfrak{so}(4,1)$-valued linear problem and we recover the Lax pair with a non-removable spectral parameter. In the second part of the paper we extend these results on the case of isothermic surfaces immersed in multidimensional Euclidean spaces. Thus we successfully checked that yet another class of integrable systems admits the group interpretation of the spectral parameter.

2. Lie point symmetries of Gauss–Codazzi equations for isothermic surfaces

Isothermic surfaces (isothermic immersions in $\mathbb{E}^3$) are defined by the property that their curvature lines admit conformal ("isothermic") parameterization [14,16]. In other words, there exist conformal coordinates such that both fundamental forms are diagonal:

$$I = e^{2\theta}(du^2 + dv^2), \qquad II = e^{2\theta}(k_1 du^2 + k_2 dv^2), \tag{1}$$

where $\theta = \theta(u,v)$ defines conformal coordinates while $k_1 = k_1(u,v)$ and $k_2 = k_2(u,v)$ are principal curvatures. These functions have to satisfy the system of Gauss–Codazzi equations:

$$\begin{aligned} &\theta_{,uu} + \theta_{,vv} + k_1 k_2 e^{2\theta} = 0,\\ &k_{1,u} + (k_1 - k_2)\theta_{,u} = 0,\\ &k_{2,v} + (k_2 - k_1)\theta_{,v} = 0. \end{aligned} \tag{2}$$

Lie point symmetries of (2) can be computed in the standard way (see [17]) applying the prolongation $\mathbf{pr}(\boldsymbol{v})$ of the vector field $\boldsymbol{v}$:

$$\boldsymbol{v} = \xi \partial_u + \eta \partial_v + \phi \partial_\theta + B_1 \partial_{k_1} + B_2 \partial_{k_2}, \tag{3}$$

where ξ, η and ϕ are functions of u, v, θ, k_1 and k_2. The prolongation $\mathbf{pr}(\boldsymbol{v})$ is computed in the standard way:

$$\mathbf{pr}(\boldsymbol{v}) = \boldsymbol{v} + \phi^u \partial_{\theta_{,u}} + \phi^v \partial_{\theta_{,v}} + \phi^{uu} \partial_{\theta_{,uu}} + \phi^{uv} \partial_{\theta_{,uv}} + B_1^u \partial_{k_{1,u}} + B_2^v \partial_{k_{2,v}},$$

where other terms are omitted because $\theta_{,vv}$, $k_{1,u}$, $k_{2,v}$ and higher derivatives can be eliminated using (2). W recall that

$$\begin{aligned} &\phi^u := D_u(\Phi - \xi\theta_{,u} - \eta\theta_{,v}) + \xi\theta_{,uu} + \eta\theta_{,uv},\\ &B_1^u := D_u(B_1 - \xi k_{1,u} - \eta k_{1,v}) + \xi k_{1,uu} + k_{1,uv},\\ &\phi^{uv} := D_{uv}(\Phi - \xi\theta_{,u} - \eta\theta_{,v}) + \xi\theta_{,uuv} + \eta\theta_{,uvv}, \end{aligned} \tag{4}$$

and analogous formulae for ϕ^u, ϕ^{uu} and B_2^v, see [17]. Finally, D_u denotes the total derivative with respect to u.

Standard symmetry procedures (see, for example [17]) yield the system of the determining equations

$$\begin{aligned} &\xi = \xi(u), \quad \eta = \eta(v), \quad \phi = \phi(\theta)\\ &B_1 = B_1(v, k_1, \theta), \quad B_2 = B_2(u, k_2, \theta),\\ &\xi_{,uu} = \eta_{,vv} = \phi_{,\theta\theta} = 0, \quad \xi_{,u} = \eta_{,v},\\ &0 = B_{1,\theta} + B_1 - B_2 + (k_1 - k_2)(\phi_{,\theta} - B_{1,k_1}),\\ &0 = B_{2,\theta} + B_2 - B_1 + (k_2 - k_1)(\phi_{,\theta} - B_{2,k_2}),\\ &0 = 2\phi - \phi_{,\theta} + 2\xi_{,u} + \frac{B_1}{k_1} + \frac{B_2}{k_2}. \end{aligned} \tag{5}$$

To obtain these we need to prolong two last equations of (2). We point out that this pair of equations is invariant with respect to simultaneous change of variables $u \leftrightarrow v$ and $k_1 \leftrightarrow k_2$. Using ordinarily understood linear independence for differential polynomials, we obtain that $B_1 = B_1(v, k_1, \theta)$, $B_2 = B_2(u, k_2, \theta)$, and $\xi = \xi(u)$, $\eta = \eta(v)$. Here also we find $\phi = \phi(\theta)$. To get the last line, and the condition $\xi_u = \eta_{,v}$, together with $\xi_{,uu} = 0 = \eta_{,vv}$, we need to use the first equation of (2) and substitute it into its prolonged version (eliminating, for instance, $\theta_{,vv}$).

Solving Eqs. (5) we get

$$\begin{aligned} &\xi = \xi(u), \quad \xi_{,uu} = 0 \;\to\; \xi = c_1 + c_0 u,\\ &\eta = \eta(v), \quad \eta_{,vv} = 0 \;\to\; \eta = c_2 + \tilde{c}_0 v,\\ &\xi_{,u} = \eta_{,v} \;\to\; \tilde{c}_0 = c_0, \quad \phi_{,\theta\theta} = 0 \;\to\; \phi = c_3 + c_5\theta,\\ &B_1 = B_1(v, k_1, \theta), \quad B_2 = B_2(u, k_2, \theta), \end{aligned} \tag{6}$$

where c_0, c_1, c_2, c_3 and c_5 are constants. Note that conditions $\xi_{,uu} = 0 = \eta_{,vv}$ follow from $\xi_{,u} = \eta_{,v}$ since each of ξ, η is a function of one variable. The last line of (5) implies that

$$B_1 = k_1 A_1(\theta, v), \quad B_2 = k_2 A_2(\theta, u) \tag{7}$$

and substituting these into the preceding two lines of (5), and once again into the last line, we obtain

$$A_2 = A_1 = c_4, \quad c_5 = 0, \tag{8}$$

where c_4 is constant. Finally, we arrive at the following result.