Physica D 342 (2017) 45–56

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Optical dispersive shock waves in defocusing colloidal media

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HIGHLIGHTS

- Analysis of an optical dispersive shock wave in a defocusing colloidal medium.
- Powerful technique to derive leading and trailing edges of dispersive shock waves.
- Near perfect predictions for the properties of the dispersive shock wave.
- Useful model for comparisons with experimental work in colloidal media.

ARTICLE INFO

Article history:
Received 14 September 2016
Accepted 17 November 2016
Available online 24 November 2016
Communicated by V.M. Perez-Garcia

Keywords:
Undular bores
Dispersive shock waves
Optical solitary waves
Modulation theory
Colloidal media

ABSTRACT

The propagation of an optical dispersive shock wave, generated from a jump discontinuity in light intensity, in a defocusing colloidal medium is analysed. The equations governing nonlinear light propagation in a colloidal medium consist of a nonlinear Schrödinger equation for the beam and an algebraic equation for the medium response. In the limit of low light intensity, these equations reduce to a perturbed higher order nonlinear Schrödinger equation. Solutions for the leading and trailing edges of the colloidal dispersive shock wave are found using modulation theory. This is done for both the perturbed nonlinear Schrödinger equation and the full colloid equations for arbitrary light intensity. These results are compared with numerical solutions of the colloid equations.

1. Introduction

Generic nonlinear wave equations, such as the Korteweg–de Vries (KdV), nonlinear Schrödinger (NLS) and Sine-Gordon equations, all possess hump-like travelling wave solutions, solitary waves [1], also referred to as solitons. However, the terms soliton and solitary wave are not strictly interchangeable, as a soliton is a solitary wave that has special properties. A solitary wave is a hump-like wave which decays to a constant level away from its peak [1]. To be termed a soliton a solitary wave must interact cleanly with other solitary waves, with the only evidence of their interaction being a possible phase change [1–3]. Another generic nonlinear wave structure is the undular bore, also called a dispersive shock wave (DSW) or collisionless shock wave [4]. The study of bores first arose in water wave theory [5,6]. Bores are the dispersive or dissipative resolution of an initial discontinuity in wave height, classic examples being the tidal bores which arise in coastal regions of strong tidal flow, such as the Severn Estuary in England and the Bay of Fundy in Canada, and the tsunamis generated by marine earthquakes and land slips. Bores in fluids fall into two broad categories, viscous bores and undular bores. As the name suggests, viscous bores are dominated by viscous loss and are steady wavetrains resulting from a balance between viscous loss, nonlinearity and dispersion [1,5,6]. Such bores with loss will not be of concern in the present work on optical DSWs. On the other hand, DSWs or undular bores arise when viscous effects are negligible and are unsteady wavetrains which spread continuously, with solitary waves at one edge and linear waves at the other. In the context of fluid flow, undular bores have been observed, studied and modelled in the atmosphere [7–9], on the continental shelf in the internal tide [10], in stratified fluids [11], in magma flow in geophysics [12–14], in Fermi gases [15] and Bose–Einstein condensates [16]. Of relevance to the present work, there have been experimental and theoretical studies of DSWs in nonlinear optical media such as photorefractive crystals [17–19], nonlinear optical fibres [20–22] and nonlinear thermal optical media [23,24]. While the terms DSW and undular bore refer to the same phenomenon and, in principle, are

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http://dx.doi.org/10.1016/j.physd.2016.11.004
0167-2789/

interchangeable, the first term will be used in the present work. This is because the term undular bore tends to be restricted to water wave theory and the term DSW is more commonly used for the phenomenon in other fields.

DSWs are unsteady wave forms and so finding solutions for them is not as straightforward as finding solutions for steady waves, such as solitary waves. It was not until the development of Whitham modulation theory [1,25,26] that a technique was developed which enabled the derivation of DSW solutions of suitable nonlinear wave equations. Whitham modulation theory is a method to analyse slowing varying periodic wavetrains using either a Lagrangian formulation of the equations or conservation equations [1], and is related to the method of multiple scales in asymptotic analysis. For this reason it is sometimes referred to as the method of averaged Lagrangians. It is also a nonlinear extension of the WKB method. The modulation equations derived using Whitham modulation theory are equations for the slowly varying parameters of the wavetrain, such as amplitude, wavenumber and mean height. If the underlying wavetrain is stable, then the modulation equations form a hyperbolic system and if it is unstable, the modulation equations form an elliptic system [1]. In particular, Whitham derived the modulation equations for the KdV equation, which were found to form a hyperbolic system [1,26]. It was subsequently realised that a simple wave (expansion fan) solution of these modulation equations was physically a DSW solution [27,28]. With this connection with Whitham modulation theory, DSW solutions could be derived for other nonlinear wave equations, such as the NLS equation [29], the Sine-Gordon equation [30] and the Gardner equation [31]. However, finding these DSW solutions as simple wave solutions relied on setting the hyperbolic modulation equations in Riemann invariant form, which is only guaranteed if the underlying nonlinear wave equation is integrable [32]. Recently, El [4,33,34] showed that, in general, hyperbolic modulation equations have a simplified structure at the leading and trailing edges of a DSW. This simplified structure was then exploited to determine its leading and trailing edges without a full knowledge of the Whitham modulation equations for the governing equation. For negative dispersion, the leading edge consists of solitary waves and the trailing edge linear waves, with the position of these waves swapped around for positive dispersion. This relaxation of the need for the full modulation equations then enabled the leading and trailing edges of DSWs governed by non-integrable equations to be determined [4,11,14,35,36]. In many observational measurements only the solitary wave edge of an DSW can be resolved [7–10,23,37], so the restriction of El's method to the leading and trailing edges of a DSW is less critical than may first appear.

In the present work the propagation of an optical DSW in the nonlinear optical medium of a colloidal suspension will be studied. The equations governing optical beam propagation in a colloid consist of an NLS-type equation for the beam coupled to an algebraic equation for the concentration of the colloid particles which depends on the beam intensity [38,39]. In the limit of low light intensity, these equations can be asymptotically reduced to a higher order NLS equation. While a colloid is normally a focusing medium, so that its refractive index increases with beam intensity, it can be made to be a defocusing medium [40,41], which then supports a DSW consisting of dark solitary waves at the trailing edge and linear waves at the leading edge [4,29,42]. The DSW is generated by a jump initial condition in optical beam intensity. While there have been previous studies of DSW in colloids [43,44], these have been for focusing colloids. In this case the waves of the DSW are modulationally unstable, so that the DSW structure has only a finite propagation length before becoming unstable. This is not the case for a defocusing colloidal medium. The leading and trailing edges of the colloid DSW are determined using El's method [4,33,34] based on both the full colloid equations and their limit in terms of a higher order NLS equation in the limit of low beam intensity. These modulation theory solutions are compared with full numerical solutions of the governing colloid equations. As well as determining the accuracy of modulation theory, these comparisons also determine the applicability of the low light intensity limit of the colloid equations.

2. Colloid equations

Let us consider the propagation of a polarised optical beam through a colloidal suspension. DSWs in nonlinear optical media are governed by NLS-type equations and, in the simplest approximation, are governed by $(1 + 1)$ dimensional equations [4,45]. Higher dimensional $(2 + 1)$ dimensional DSWs governed by NLS-type equations are much more difficult to analyse and need a nontrivial azimuthal vortex structure to be stable. Indeed, solutions of $(2 + 1)$ dimensional DSWs governed by not just NLS-type equations, but any nonlinear wave equation, are an open topic [46,47]. Hence, the optical beam generating the colloid DSW will be assumed to have a plane front. The z direction will then be taken to be the propagation direction, with the x direction orthogonal to this and the beam having no y dependence. The concentration of the colloid particles has a nonlinear dependence on the beam intensity. The colloid can be either a focusing medium, so that its refractive index increases with beam intensity [38,48], or defocusing, so that it decreases with intensity [40,41]. In order for a stable DSW to be generated, the colloidal medium will be assumed to be defocusing. Let us denote the concentration of the colloidal particles by η, with η_0 the constant background concentration in the absence of the optical beam. In the slowly varying, paraxial approximation the non-dimensional equations governing the propagation of the optical beam through the colloidal suspension are then [38,39]

$$i\frac{\partial u}{\partial z} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - (\eta - \eta_0)u = 0, \tag{1}$$

with the equation of state, that is the medium response equation,

$$|u|^2 = g(\eta) - g_0, \quad g(\eta) = \frac{3-\eta}{(1-\eta)^3} + \ln\eta. \tag{2}$$

Here u is the complex valued envelope of the electric field of the optical beam and $g_0 = g(\eta_0)$. The Carnahan–Starling compressibility approximation has been used for the state relation g. Alternative models for the compressibility alter the form of g. The Carnahan–Starling approximation is valid up to the solid–fluid transition, which occurs at $\eta = \sqrt{2}\pi/9 \approx 0.496$ in a hard-sphere fluid [49]. It should be noted that the nonlinear term in the NLS equation (1) for the optical beam has a negative coefficient, so that the equation is defocusing, in contrast to the focusing equation of previous work [38,39].

Hoefer [50] considered general properties of DSW solutions of generalised NLS equations of the form

$$i\frac{\partial u}{\partial z} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - f(|u|^2)u = 0, \tag{3}$$

with details determined for a power law nonlinearity f. While the colloid system (1) and (2) is in principle of this form, the nonlinearity f cannot be explicitly determined from the medium response equation (2). The colloid system then represents a further extension of the forms of nonlinear response in generalised NLS equations and the DSW solutions for such equations, in addition to the previously studied extension of a nonlocal response [35,51].

The simplest initial condition which will result in the generation of a DSW is a step initial condition in optical intensity,

$$u(x,0) = \begin{cases} u_-, & x < 0, \\ u_+, & x > 0. \end{cases} \tag{4}$$