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Nonlinear eigenvalue problems of the elastica ☆

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ABSTRACT

Physical phenomena are usually described by nonlinear differential equations. If some of the physical parameters are unknown then adding appropriate constraints may transform a nonlinear problem to a nonlinear eigenvalue problem. This principle is illustrated by a variety of problems associated with deflections of a flexible rod.

The basic problem studied is the problem formed by the natural extension of the linear buckling problem of a flexible rod when large deflections are taken into account. The unknown parameter in these problems is the load, and the constraint is the predetermined distance between the rod's ends. The problem of finding large deflections of the strongest column is also addressed. A simple numerical method for solving these problems is given.

The paper gives insight into the problem of forming a non-linear eigenvalue problem. It highlights the relations between the mathematical formulation and the governing physical laws, and demonstrates the similarity between the linear and the nonlinear solutions.

1. Introduction

Consider an inextensible thin uniform rod of length l and flexural rigidity p. The rod is bent by two collinear opposing forces $R=\mu p$ applied at its ends. The problem of inflexional elastica, studied by Euler, may be formulated as follows:

Problem A. Given: μ, and ϕ_0, $0 \le \phi_0 < \pi$
Find: l, and $\phi(s)$, $0 \le s \le l$, satisfying

$$\begin{cases} \frac{d^2\phi}{ds^2} + \mu \sin \phi = 0 & 0 < s < l \\ \phi(0) = \phi_0 \quad \phi'(0) = 0 \quad \phi'(l) = 0 \end{cases} \tag{1}$$

where s is the arc-length coordinate of the rod and $\phi(s)$ is the slope at s.

The solution to this problem, given by Love [19], pp. 402–403, is

$$\phi = 2\arcsin(k\, sn(s\sqrt{\mu} + K)), \quad 0 \le s \le l, \quad l = 2iK(k), \quad i = 1, 2, 3, \ldots \tag{2}$$

where

$$k = \sin \frac{\phi_0}{2}, \tag{3}$$

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$K(k)$ is the complete elliptic integral of the first kind,

$$K=\int_0^{\pi/2}\frac{du}{\sqrt{1-k^2\sin^2u}}, \tag{4}$$

and $\mathrm{sn}(u)$ is the Jacobian elliptic sine of argument u and of modulus k, see Abramowitz and Stegun [1] for standard definitions. Frisch-Fay [7] gives analytical solutions to many related problems of large deflections of flexible rods.

Eq. (2) defines the family of inflectional elastica. For example, with $\phi_0=3\pi/5$ and $\mu=20$, the first four solutions of Problem A are shown in Fig. 1. The configuration corresponding to the smallest length of the rod, shown in Fig. 1a, is the fundamental configuration that defines all other curves in the family. The line connecting the end points of the rod is the line of action for the applied forces. The intersections of the curve with the line of action, e.g., points A, B and C, in Fig. 1b, define the inflexional points, the points of vanishing curvature. Segments of the elastica may define solutions to additional problems. For example the segment PB of the curve in Fig. 1b is the shape of a rod that is clamped at P and bent by a vertical force of magnitude $4R$ applied at B. By combining various segments of the elastica in a continuous and smooth manner a variety of additional problems may be solved, as done in Frisch-Fay [7].

We note in passing that while the solution given by Eqs. (2) through (4) is simple, numerical solution of Problem A may be expressed by a simple explicit non-iterative three-term recursion involving elementary functions only.

By using mathematical considerations alone one could not say whether Problem A is a boundary- or initial-value problem. The problem has enough conditions at $s=0$ to be considered an initial value problem and it has enough conditions at $s=0$ and $s=l$ to be considered a boundary value problem. From physical point of view it is clear that Problem A is a boundary value problem that depends on two boundary conditions. The added condition, $\phi(0)=\phi_0$, may be regarded as a constraint. However, should s be the time, and μ be a gravity constant, Problem A would be interpreted as the problem of finding the half-period time l_1, and its integer multiplication, of a unit length pendulum that is released from an angle ϕ_0 at time $s=0$. This observation leads to the Kirchhoff's kinetic analogue stated by Love [19], p. 399, as follows:

> *"The equation of equilibrium of a thin rod, straight and prismatic when unstressed, and held bent and twisted by forces and couples applied at its ends alone, can be identified with the equations of motion of a heavy rigid body turning about a fixed point"*

With this analogy Problem A may be regarded as an initial value problem, where $\phi'(l)=0$ is the constraint.

Since $\phi(s)$ varies between ϕ_0 and $-\phi_0$ for all s, it follows that when $\phi_0\to 0$ the slope everywhere is small. The central line coordinate s may be replaced in this case by the x-axis of a Cartesian coordinate system that is attached to the ground and

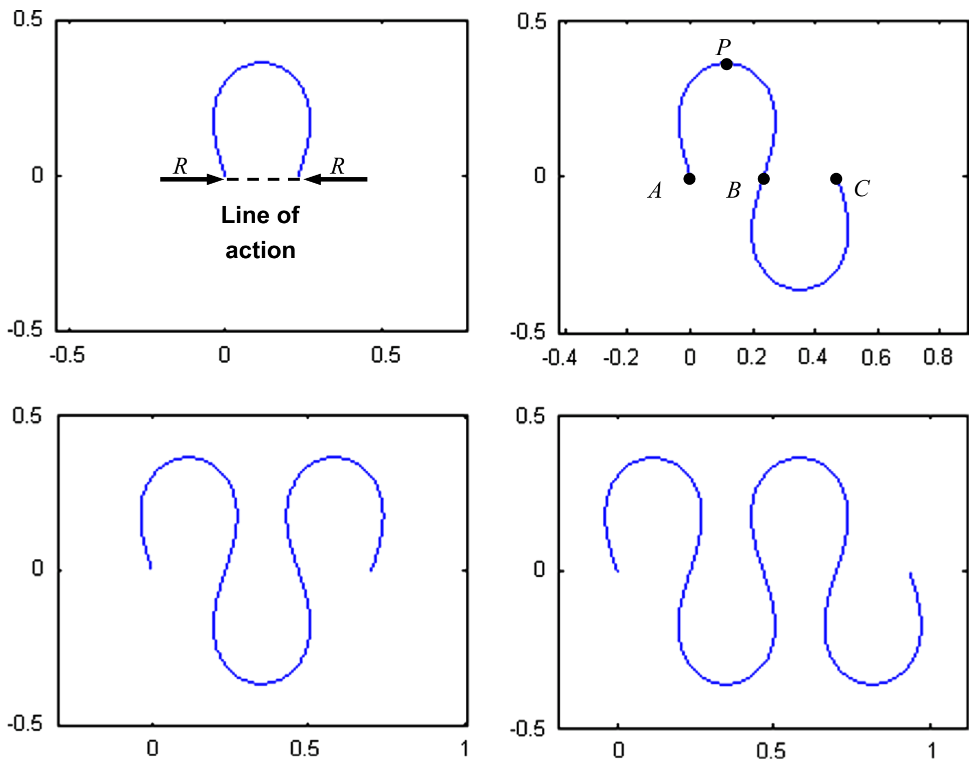

Fig. 1. The first four curves in the family of inflexional elastica, $\mu=R/p=20$, $\phi(0)=3\pi/5$.