Mechanical Systems and Signal Processing 45 (2014) 563–578

Contents lists available at ScienceDirect

Mechanical Systems and Signal Processing

journal homepage: www.elsevier.com/locate/ymssp

# Nonlinear power flow analysis of the Duffing oscillator

J. Yang, Y.P. Xiong*, J.T. Xing

*Faculty of Engineering and the Environment, Fluid Structure Interactions Research Group, University of Southampton, Highfield, Southampton SO17 1BJ, UK*

ARTICLE INFO

*Article history:*
Received 24 February 2013
Received in revised form
8 November 2013
Accepted 9 November 2013
Available online 13 December 2013

*Keywords:*
Power flow analysis
Duffing oscillator
Nonlinear stiffness
Super-/sub-harmonic resonances
Bifurcation
Chaos

ABSTRACT

Power flow characteristics of different forms of the Duffing oscillator, subject to harmonic excitations, are studied in this paper to reveal the distinct power input and dissipation behaviour arising from its nonlinearity. Power flow variables, instead of the displacement and velocity responses, are used to examine the effects of nonlinear phenomena including sub-/super-harmonic resonances, non-uniqueness of solutions, bifurcations and chaos. Both analytical harmonic balance approximations and Runge–Kutta numerical integrations are adopted to effectively address instantaneous/time-averaged power flows of the system with periodic/chaotic motions without losing the essential nonlinear characteristics. It is demonstrated that only the in-phase velocity component with the same frequency as the excitation contributes to the time-averaged input power (TAIP). It is shown that super-/sub-harmonic resonances may result in substantial increases in TAIP and the nonlinearity leads to varying time-averaged power flow levels sensitive to the initial conditions. The study reveals that bifurcations may cause large jumps in time-averaged input power. However, for bifurcations of periodic to chaotic motions encountered in the low-frequency range, the corresponding variations in TAIP of the double-well potential systems are small. For a chaotic response, the associated TAIP is insensitive to the initial conditions but tends to an asymptotic value as the averaging time increases, and thus can be used as a measure to quantify chaotic responses. The paper concludes some inherently nonlinear power flow characteristics which differ greatly from those of the linear systems, and provides useful information for applications.

## 1. Introduction

Vibrational power flow analysis (PFA) approach has become a widely accepted tool to investigate dynamic behaviour of coupled systems and complex structures. Compared with individual measures such as force and displacement transmissibility, vibrational power flow combines the effects of force and velocity amplitudes as well as their relative phase angle in a single quantity, and thus can better reflect the transmission of vibration energy between various sub-systems of an integrated structure. The PFA fundamental concepts were discussed by Goyder and White [1]. In recent years, various approaches, such as a dynamic stiffness method [2], a receptance method [3], a mobility method [4], a wave intensity method [5], a finite-element based energy flow modelling technique [6] and progressive approaches [7] were developed and applied to investigate vibration control systems [8]. Instead of investigating individual structures such as coupled beam/plate-like structures or periodic systems, Xing and Price [9] proposed a more general PFA approach based on the fundamental

---

* Corresponding author. Tel.: +44 2380596619; fax: +44 2380 597744.
*E-mail addresses:* y.xiong@soton.ac.uk, y.p.xiong@ship.soton.ac.uk (Y.P. Xiong).

0888-3270/$ - see front matter Crown Copyright © 2013 Published by Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.ymssp.2013.11.004

principles of continuum dynamics, in which the energy-flow vector, energy-flow potential and energy-flow lines describing energy flow distributions were defined. Xiong et al. [10] developed a power flow mode theory to reveal the inherent power flow behaviour of a dynamic system based on its damping distribution, from which energy flow design approaches were proposed to achieve specific power flow patterns satisfying vibration control requirements.

Although significant advances on power flow characteristics of linear dynamical systems have been reported, investigations on power flows of nonlinear vibrating systems are limited. However, dynamic systems in practice are inherently nonlinear, and strong nonlinear effects have been encountered in many applications. For example, nonlinear models for both damping and stiffness were needed for accurate dynamic analysis of hydraulic engine mounts used in the automobile industry [11]. The damping coefficient of orifice-type dampers varies with their internal geometry, frequency of flow oscillation and the Reynolds number [12]. There may also be significant nonlinearity in structural joints [13]. Moreover, introducing nonlinear elements to a design may bring benefits which could not be achieved by linear systems. For instance, it has been shown that nonlinear vibration absorbers can successfully enlarge the effective working frequency range of their linear counterparts [14]. A nonlinear negative stiffness mechanism can be used in parallel with linear isolators to improve the effectiveness of vibration isolators in the low-frequency range [15].

In view of these facts, there has been a growing interest in studying nonlinear dynamical systems from the perspective of power flows in past few years. Royston and Singh [16] employed vibratory power transmission as a performance index in the optimisation of multiple degrees-of-freedom nonlinear mounting systems, and examined automotive hydraulic engine mounts by investigating vibratory power flows from an excited rigid body through a nonlinear path into a resonant receiver [17]. Xiong et al. [18] studied a nonlinear coupling system consisting of a machine, a generic nonlinear isolator and a flexible beam-like ship travelling in seaway. The nonlinearity was characterised by a $p$th power damping term and a $q$th power stiffness term, and was shown to have a significant influence on the system's power flows, especially when the excitation frequency is close to resonant frequencies. Oscillators with essentially nonlinear stiffness may exhibit the phenomenon of targeted energy transfer (TET), which corresponds to one-way channelling of the vibrational energy from a primary structure to a passive nonlinear attachment [19]. Based on the time-averaged input power information associated with free oscillations of conservative systems, a frequency-energy plot (FEP) can be used to represent nonlinear normal modes and the frequency-energy dependence of nonlinear systems [20]. Yang et al. [15] used time-averaged power flow quantities instead of traditional force transmissibility to assess vibration isolation performance and attempted to quantify the nonlinear responses of the Duffing oscillator by power flow analysis [21].

Previous research has clearly shown that a better understanding of power flow patterns in nonlinear dynamical system can bring valuable benefits for science and engineering. However, due to a lack of power flow theory and effective modelling and simulation methods to deal with systems involving complex nonlinear phenomena, the influences of nonlinearity on system power flows remain unclear. Fundamental studies are still needed to reveal the basic principles governing vibration power generation, dissipation and transmission in nonlinear dynamical systems.

In this paper, the power flow behaviour of a typical nonlinear system, the Duffing oscillator, is investigated as an attempt to address the above problem. This system has been extensively studied with focus on its displacement/velocity responses characteristics [22–24]. Nevertheless, new information can still be obtained by examining it from another perspective of vibrational power flows. Such examination is necessary considering that the influence of the stiffness nonlinearity on power flows has not been clarified and also the findings may provide promising applications to vibration control and energy harvesting. Emphasis of the present study will be placed on revealing the associated power flow behaviour of the system when it exhibits complex nonlinear phenomena, such as sub-/super harmonic resonances, bifurcations and chaotic motions. Following the derivations power flow formulations, the solution methods used in the paper are briefly described. The harmonic balance method is used for analytical approximations of the power flow variables of the system undergoing periodic motions. Numerical simulations are conducted to investigate the instantaneous power flows, to verify the analytical approximations and to examine the effects of chaotic motion on system power flows. Conclusions and some suggestions for applications are provided at the end of the paper.

## 2. Power flow formulations and solution approaches

### 2.1. Power flow formulations

The Duffing oscillator is governed by the equation

$$\ddot{x}+2\xi\dot{x}+\alpha x+\beta x^3=f\ \cos\ \omega t, \tag{1}$$

in which the restoring force is characterised by a linear term $\alpha x$ and a cubic nonlinear term $\beta x^3$. It may be referred to as a softening stiffness system (Case I) when $\alpha>0,\ \beta<0$; a hardening stiffness system (Case II) when $\alpha>0,\ \beta>0$; or a double-well potential system (Case III) when $\alpha<0,\ \beta>0$. A system with $\alpha<0,\ \beta<0$ has non-positive stiffness, thus it is unstable and will not be investigated in this paper.

Multiplying by the velocity $\dot{x}$ on both sides of Eq. (1), we derive the power flow balance equation of the system in the form:

$$\dot{K}+\dot{U}+p_d=p_{in}, \tag{2}$$