Signal Processing 131 (2017) 36–48

Contents lists available at ScienceDirect

Signal Processing

journal homepage: www.elsevier.com/locate/sigpro

A simultaneous sparse approximation method for multidimensional harmonic retrieval ☆

Souleymen Sahnoun [a,*], El-Hadi Djermoune [b], David Brie [b], Pierre Comon [a]

[a] CNRS, Gipsa-Lab, Univ. Grenoble Alpes, F-38000 Grenoble, France
[b] CRAN, Université de Lorraine, CNRS, Boulevard des Aiguillettes, BP 239, 54506 Vandoeuvre Cedex, France

ARTICLE INFO

Article history:
Received 26 February 2016
Received in revised form
31 May 2016
Accepted 25 July 2016
Available online 28 July 2016

Keywords:
Multidimensional harmonic retrieval
Frequency estimation
Simultaneous sparse approximation
Multigrid dictionary refinement
Cramér–Rao lower bound

ABSTRACT

In this paper, a new method for the estimation of the parameters of multidimensional (R-D) harmonic and damped complex signals in noise is presented. The problem is formulated as R simultaneous sparse approximations of multiple 1-D signals. To get a method able to handle large size signals while maintaining a sufficient resolution, a multigrid dictionary refinement technique is associated to the simultaneous sparse approximation. The refinement procedure is proved to converge in the single R-D mode case. Then, for the general multiple modes case, the signal tensor model is decomposed in order to handle each mode separately in an iterative scheme. The proposed method does not require an association step since the estimated modes are automatically "paired". We also derive the Cramér–Rao lower bounds of the parameters of modal R-D signals. The expressions are given in compact form in the single tone case. Finally, numerical simulations are conducted to demonstrate the effectiveness of the proposed method.

1. Introduction

The problem of estimating the parameters of sinusoidal signals from noisy measurements is an important topic in signal processing and several parametric and nonparameteric approaches have been developed for one-dimensional (1-D) signals [1]. Recently, this problem has received a renewed interest thanks to the emergence of multidimensional (R-D) applications. Indeed, parameter estimation from R-D signals is required in numerous applications in signal processing and communications such as nuclear magnetic resonance (NMR) spectroscopy, wireless communication channel estimation [2] and MIMO radar imaging [3]. In all these applications, signals are assumed to be a superposition of R-D sinusoids or, more generally, of exponentially decaying R-D complex exponentials (modal signals). As for the 1-D case, the crucial step is the estimation of the R-D modes (including frequencies and damping factors) because they are nonlinear functions of the data. In this paper, we consider the single snapshot R-D signal model described in [4].

In order to achieve high resolution estimates, parametric approaches are often preferred to nonparametric ones. Several parametric R-D methods ($R \geq 2$) have been proposed. They include linear prediction-based methods such as 2-D TLS-Prony [5], and subspace approaches such as matrix enhancement and matrix pencil (MEMP) [6], 2-D ESPRIT [7], multidimensional folding (MDF) [8], improved multidimensional folding (IMDF) [9,10], Tensor-ESPRIT [11], principal-singular-vector utilization for modal analysis (PUMA) [12,13] and the methods proposed in [14,15]. All these methods perform at various degrees but it is generally admitted that they yield accurate estimates at high SNR scenarios and/or when the frequencies are well separated. This is obtained at the expense of computational effort. In [12], tensor PUMA was proposed as an accurate and computationally efficient multidimensional harmonic retrieval method, which attains the Cramér–Rao lower bound (CRLB) and does not require to build large size matrix or tensor. However its performance degrades rapidly with the increase of the number of components in the R-D signal.

Recently, methods based on sparse approximations have been proposed to address the harmonic or modal retrieval problem [16–23]. For time-data spectral estimation, the dictionary is formed from a set of (normalized) complex exponentials potentially embedded in the data, which allows one to easily include some prior knowledge about the position of some known modes. More generally, the usual choice is a uniform spectral grid obtained by sampling the frequency and damping factor lines. Clearly, a fine

☆ This work is funded by the European Research Council under the European Community's Seventh Framework Programme FP7/2007–2013 Grant Agreement no. 320594, DECODA project.

* Corresponding author.
E-mail addresses: souleymen.sahnoun@gipsa-lab.grenoble-inp.fr (S. Sahnoun), el-hadi.djermoune@univ-lorraine.fr (E.-H. Djermoune), david.brie@univ-lorraine.fr (D. Brie), pierre.comon@gipsa-lab.grenoble-inp.fr (P. Comon).

http://dx.doi.org/10.1016/j.sigpro.2016.07.029
0165-1684/

grid is required to get a good resolution but, on the other hand, it will result in a huge dictionary [16]. This complexity is further increased in the case of R-D signals in which we are confronted with $2R$-D grids. In order to reduce the computational burden, a multigrid scheme for sparse approximation was proposed in [20] to iteratively refine the dictionary starting from a coarse one. At each iteration, a sparse approximation is performed and then new grid points (called "atoms") are inserted in the vicinity of active ones leading to a multiresolution-like scheme. This algorithm, which refines jointly R2-D grids, is efficient but has mainly two drawbacks: (1) it does not have convergence guarantees, (2) the dictionary becomes intractable for large signals when $R \geq 2$. Recently, several studies have also focused on gridless sparse recovery methods based on continuous dictionaries [24,25]. However, the proposed algorithms demand a large computational burden even for 1-D signals.

The goal of the present paper is to propose a fast multidimensional modal estimation technique able to handle large signals and yielding a good estimation accuracy.

1. First, the proposed approach, as for some parametric methods for modal retrieval, is based on the idea of estimating the parameters independently along each dimension $r = 1, \ldots, R$. It will be shown that the *simultaneous* sparse approximation concept [26,21] is well-suited for R-D modal retrieval ($R \geq 2$).
2. The second contribution consists in the proposition of a new multigrid scheme which amounts to consider a two-step refinement of 1-D grids, the first step for frequencies and the second one for damping factors. One advantage of this procedure is that it reduces the computational time. The convergence of the proposed multigrid strategy is analyzed in the single tone case ($F = 1$), and convergence conditions are expressed in terms of atom positions in the initial dictionaries.
3. The extension of this result to the multiple tones case ($F > 1$) is not trivial because, not only it depends on the selected sparse approximation algorithm, but also on the coherence of the dictionary [26]. Indeed, due to the multigrid strategy, the columns of the refined dictionary are increasingly correlated, which may prevent convergence even in the noiseless case. Consequently, for $F > 1$, we exploit an alternative representation of the data model enabling the extraction of the R-D signal tones separately. Therefore, the third contribution of this paper is the derivation of a new algorithm for estimating parameters of R-D damped signals in which the results of the previous contribution apply. The effectiveness of the new algorithm for multiple R-D tones is also analyzed. One very interesting by-product of this approach is that the pairing of R-D parameters is achieved for free, without any further association stage.

The usual way to assess the performances of an estimation method is to compare the variance of the estimates to the CRLB. In [6] Hua derived the CRLB for 2-D frequencies, i.e., undamped 2-D exponentials; no damped signals are considered. Closed-form expressions of the CRLB for the general undamped R-D case are derived in [27]. CRLB for 2-D damped signals are derived in [28]. Therefore, to the best of our knowledge, no compact expressions of the CRLB's are available for the general R-D damped model. Thus, another contribution of the paper is the derivation of the CRLB's for the frequency, damping factor, amplitude and phase of this model.

The remainder of this paper is organized as follows. In Section 2, we introduce notation and present the R-D modal retrieval problem. In Section 3, we formulate the R-D modal estimation problem as R simultaneous sparse estimation problems, show how to construct a modal dictionary on a uniform grid and then describe the new fast multigrid strategy. In Section 4, we give sufficient conditions for convergence of the multigrid dictionary refinement in the case of single tone R-D signals. In light of these new results, we propose in Section 5 a new efficient algorithm for multiple tones R-D signals. In Section 6, we derive the expressions of the CRLB's for the parameters of R-D damped exponentials in Gaussian white noise. We then give the CRLB in the cases of single damped and undamped R-D cisoids. The effectiveness of the proposed method is demonstrated using simulation signals in Section 7. Finally, conclusions are drawn in Section 8.

2. Notation and problem statement

2.1. Notation

Scalars are denoted as lower-case letters (a, b, α), column vectors as lower-case bold-face letters ($\mathbf{a}$, $\mathbf{b}$), matrices as bold-face capitals ($\mathbf{A}$, $\mathbf{B}$), and tensors as calligraphic bold-face letters ($\mathcal{A}$, $\mathcal{B}$). Notations $(\cdot)^{\mathrm{T}}$, $(\cdot)^{\mathrm{H}}$ and $(\cdot)^{\dagger}$ stand for the transpose, the Hermitian transpose and the pseudo-inverse, respectively. The symbols "$\odot$" and "$\boxtimes$" will denote the Khatri–Rao product (column-wise Kronecker) and the Kronecker product, respectively. Both words "mode" and "tone" are used to refer to a component of the multidimensional signal. The tensor operations used here are consistent with [29]:

- the outer product of two tensors $\mathcal{A} \in \mathbb{C}^{M_1 \times \cdots \times M_R}$ and $\mathcal{B} \in \mathbb{C}^{K_1 \times \cdots \times K_N}$ is given by:

$$\mathcal{C} = \mathcal{A} \otimes \mathcal{B} \in \mathbb{C}^{M_1 \times \cdots \times M_R \times K_1 \times \cdots \times K_N},$$
$$c(m_1, \ldots, m_R, k_1, \ldots, k_N) = a(m_1, \ldots, m_R) b(k_1, \ldots, k_N) \tag{1}$$

- the contraction product acting on the r-th index of a tensor $\mathcal{A} \in \mathbb{C}^{M_1 \times \cdots \times M_R}$ and the second index of a matrix $\mathbf{U} \in \mathbb{C}^{K \times M_r}$ is:

$$\mathcal{B} = \mathcal{A} \underset{r}{\bullet} \mathbf{U} \in \mathbb{C}^{M_1 \times \cdots \times M_{r-1} \times K \times M_{r+1} \times \cdots \times M_R},$$
$$b(m_1, m_2, \ldots, m_{r-1}, k_r, m_{r+1}, \ldots, m_R) = \sum_{m_r=1}^{M_r} a(m_1, m_2, \ldots, m_R) u(k_r, m_r) \tag{2}$$

- the matrix $\mathbf{A}_{(r)} \in \mathbb{C}^{M_r \times (M_1 \cdots M_{r-1} M_{r+1} \cdots M_R)}$ represents the unfolding (dimension-r matricization) of the tensor $\mathcal{A}$ and corresponds to the arrangement of the dimension-r fibers of $\mathcal{A}$ in the columns of the resulting matrix.
- $\| \mathcal{A} \|$ denotes the Frobenius norm for tensors.
- The concatenation of two tensors $\mathcal{A}_1 \in \mathbb{C}^{M_1 \times \cdots \times M_{r-1} \times K_1 \times M_{r+1} \times \cdots \times M_R}$ and $\mathcal{A}_2 \in \mathbb{C}^{M_1 \times \cdots \times M_{r-1} \times K_2 \times M_{r+1} \times \cdots \times M_R}$ along the rth dimension is denoted by $\mathcal{A}_1 \sqcup_r \mathcal{A}_2$ and obtained by stacking $\mathcal{A}_1$ and $\mathcal{A}_2$ along the rth dimension.

Finally, throughout this paper, the tilde symbol ($\tilde{}$) denotes a noisy signal; e.g. $\tilde{y}(\cdot) = y(\cdot) + e(\cdot)$.

2.2. Problem formulation

An R-D modal signal is modeled as the superposition of F multidimensional damped complex sinusoids:

$$\tilde{y}(m_1, \ldots, m_R) = \sum_{f=1}^{F} c_f \prod_{r=1}^{R} a_{f,r}^{m_r - 1} + e(m_1, \ldots, m_R) \tag{3}$$

where $m_r = 1, \ldots, M_r$ for $r = 1, \ldots, R$. M_r denotes the sample support of the r-th dimension, $a_{f,r} = \exp(\alpha_{f,r} + j\omega_{f,r}) \in \mathbb{C}$ is the f-th mode in the r-th dimension, $\{\alpha_{f,r}\}_{f=1,r=1}^{F,R}$, $\alpha_{f,r} \in \mathbb{R}^{-}$, are the damping factors, $\{\omega_{f,r} = 2\pi\nu_{f,r}\}_{f=1,r=1}^{F,R}$ are the angular frequencies, and $c_f = \lambda_f \exp(j\phi_f)$ is the complex amplitude of the f-th mode where