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Critically sampled graph filter banks with polynomial filters from regular domain filter banks

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ABSTRACT

Graph signal processing deals with the processing of signals defined on irregular domains and is an emerging area of research. Graph filter banks allow the wavelet transform to be extended for processing graph signals. Sakiyama and Tanaka (2015) [22] recently proposed a technique to convert linear-phase biorthogonal filter banks for regular domain signals to biorthogonal graph filter banks. Perfect reconstruction is preserved using the technique but the resulting spectral filter functions are transcendental and not polynomial. Polynomial function filters are desired for the localization property and implementation efficiency. In this work we present alternative techniques to perform the conversion. Perfect reconstruction is preserved with the proposed techniques and the resulting spectral filters are polynomial functions.

1. Introduction

There are many applications, e.g. social, biological and sensor networks, where the data is defined on an irregular domain. Irregular domains are best modelled using graph theory techniques and this has spawned great interest among signal processing researchers to develop graph signal processing (GSP) techniques. Excellent recent reviews of this fast emerging area can be found in [1,2]. Some applications where the signal is naturally defined in the regular-domain, e.g. images, can also benefit from GSP techniques [1]. The extension of techniques from regular-domain signal processing to processing graph signals are non-trivial and sometime non-unique [1,2]. The notion of frequency and frequency domain is perhaps one of the most important and useful concepts in regular-domain signal processing. An extension of this notion that is based on the adjacency matrix is presented in [3,2]. Spectral graph theory [4], which is based on the Laplacian matrix, provides a natural extension of the notion of frequency and frequency domain for graph that are undirected [5,1].

The wavelet transform is without doubt one of the most powerful tools for processing regular-domain signals in many applications [6–9]. Several researchers have proposed extending the wavelet transform for graph signals [10–17,5]. The graph wavelet designs can broadly be classified into either the vertex domain designs (analogous to the spatial domain) or the spectral domain designs (analogous to the frequency domain). The spectral domain design relies on the use of the graph Laplacian to give a spectral representation that is similar to the Fourier transform for regular-domain signals. The wavelet transform that is based on the two-channel multirate filter bank is one of the most popular type of regular-domain transform [18]. Most of the graph transforms proposed however are not based on the critically sampled filter bank. Using spectral graph theory, the two-channel critically sampled filter bank was extended by Narang and Ortega for processing signals in the graph domain [15,16]. The key to achieving critical sampling is the introduction of the downsampling/upsampling operation on bipartite graphs. Any arbitrary graph can be decomposed into a series of bipartite graphs. By sequential application to this series of bipartite graphs, the basic two channel bipartite filter bank can be used to implement a wavelet transform for signals defined over any arbitrary graph. This is similar to the notion of separable filtering for multidimensional regular-domain signals. The design in [15] is orthogonal and the design in [16] is biorthogonal. For achieving localization and efficient implementation, the spectral filter must be a finite polynomial of the spectral variable (usually denoted by λ). Perfect reconstruction can be achieved with polynomial spectral filters for the biorthogonal case but cannot be achieved for the orthogonal case. The critically sampled graph wavelet transform has been applied to the nonlinear approximation of signals in [15,16]. It was shown in [15,16] that the graph wavelets gave better results than regular-domain wavelets for images as the edge information was better captured with the former. The biorthogonal graph wavelet transform has

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also recently been applied to the compression of human body sequences [19,20] with significant performance improvement over direct encoding using the state-of-the-art video encoder HEVC (High Efficiency Video Coder).

The design of regular-domain filter banks (RDFB) is a mature area and there is a plethora of design techniques and filter coefficients in the research literature. For example the Matlab wavelet toolbox has functions to generate the Cohen–Daubechies–Feaveau (CDF) wavelet filter coefficients [21]. The recent work in [22] developed techniques for converting real-valued critically sampled RDFB to graph filter banks (GFB). This is an alternative to the direct design of GFB [15,16,23–25]. The techniques in [22] involves a direct linear mapping of the frequency variable (usually denoted by ω) of the RDFB to the spectral variable (usually denoted by λ) of the GFB. The resulting spectral filters are transcendental (trigonometric) functions and are not polynomials. For localization and efficient implementation a polynomial approximation to the transcendental function is used and perfect reconstruction (PR) can only be approximated. In this paper a different approach is proposed to convert RDFB to GFB. The resulting spectral filters are polynomial functions and exact PR is preserved in the GFB.

An overview of the paper is as follows. A review of the fundamentals of graph signal processing is presented in Section 2. Section 3 presents the new conversion techniques and proves certain associated results. The conversion relies on the construction of certain functions and Section 4 discusses the choices for these functions. Design examples are presented in Section 5. Conclusions are given in Section 6.

2. Preliminaries

A brief review of graph theory, graph signal processing and biorthogonal graph filter banks is presented here. Results are stated without proofs. More details can be found in [5,15,1,16].

2.1. Spectral graph theory

A graph G is a mathematical object consisting of vertices and edges. A graph can be directed or undirected but only the latter will be considered in this paper. The set of vertices V and the set of edges E defines the graph $G=(V, E)$. The number of vertices is denoted as $N=|V|$ and the vertices are labelled as $1, \ldots, N$. An edge $e\in(i, j)\in E$ connects the two vertices i and j. The adjacency matrix $\mathbf{A}$ is the $N\times N$ symmetric matrix whose element $a_{i,j}(i, j=1, \ldots, N)$ is positive real and gives the weight of the edge connecting vertices i and j. If there is no edge connecting vertices i and j then $a_{i,j}=0$. Only graphs with no self loops $(a_{i,i}=0)$ are considered in this paper. An example of a graph with N=7 vertices and 6 edges is shown in Fig. 1. The vertex i degree is $d_i\equiv\sum_j a_{i,j}$ and the diagonal matrix $\mathbf{D}$ is $\mathbf{D}\equiv\mathrm{diag}(d_i)$. The unnormalized (combinatorial) graph Laplacian matrix is defined as $\mathbf{L}\equiv\mathbf{D}-\mathbf{A}$. The *normalized* graph Laplacian, with respect to $\mathbf{D}$, is defined as [5,15,1,16]

$$\mathcal{L}\equiv\mathbf{D}^{-1/2}\mathbf{L}\mathbf{D}^{-1/2}=\mathbf{I}-\mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}$$

where $\mathbf{I}$ is the identity matrix. Only the normalized Laplacian $\mathcal{L}$ will be considered in this paper. Since $\mathcal{L}$ is a real symmetric matrix, it can be decomposed as [5,15,1,16]

$$\mathcal{L}=\sum_{i=1}^{N}\lambda_i\mathbf{u}_i\mathbf{u}_i^T=\mathbf{U}\mathbf{\Lambda}\mathbf{U}^T$$

where $\mathbf{\Lambda}=\mathrm{diag}(\lambda_i)$ and $\mathbf{U}\equiv[\mathbf{u}_1|\mathbf{u}_2|\cdots|\mathbf{u}_N]$, with λ_i being the eigenvalue of $\mathcal{L}$ and $\mathbf{u}_i$ being the corresponding eigenvector. Now $\mathbf{U}\mathbf{U}^T=\mathbf{I}$ (orthogonal matrix), so the eigenvectors $\mathbf{u}_1, \cdots, \mathbf{u}_N$ form an orthonormal set. The set of eigenvalues $\sigma(G)\equiv\{\lambda_1\le\lambda_2\cdots\le\lambda_N\}$ is the spectrum of graph G. The eigenvalues are bounded in the interval $[0, 2]$ and correspond to the graph natural oscillation frequencies.

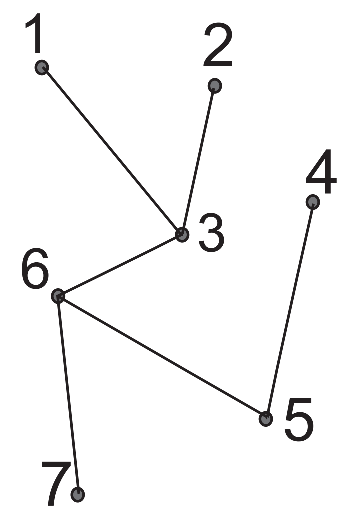

Fig. 1. An example of a graph with 7 vertices and 6 edges.

2.2. Graph signal filtering

A signal over a graph G is a function that maps each vertex i to a numerical value $f(i)$. The graph signal can be represented as the vector $\mathbf{f}=[f(1)\cdots f(N)]^T$. The graph Fourier transform is defined as [5,15,1,16]

$$\hat{f}(\lambda_l)\equiv\sum_{n=1}^{N}f(n)u_l(n)=\mathbf{f}^T\mathbf{u}_l$$

where $\mathbf{u}_l\equiv[u_l(1)\cdots u_l(N)]^T$ (for $l=1, \cdots, N$) are the Laplacian eigenvectors. The equation can be compactly written as $\hat{\mathbf{f}}=\mathbf{U}^T\mathbf{f}$ where $\hat{\mathbf{f}}=[\hat{f}(\lambda_1)\cdots\hat{f}(\lambda_N)]^T$ is the vector of the spectral component at the graph frequencies. The inverse graph Fourier transform is therefore $\mathbf{f}=\mathbf{U}\hat{\mathbf{f}}$ or in scalar form $f(n)=\sum_{l=1}^{N}\hat{f}(\lambda_l)u_l(n)$. Filtering in the spectral domain is defined as $\hat{f}_{out}(\lambda_l)=h(\lambda_l)\hat{f}(\lambda_l)$ where $h(\lambda)$ is the spectral filter in the continuous spectral variable λ. Taking the inverse transform of $\hat{f}_{out}(\lambda_l)$ gives the output in the vertex domain as $f_{out}(n)=\sum_{l=1}^{N}\hat{f}(\lambda_l)h(\lambda_l)u_l(n)$. In vector/matrix form $\mathbf{f}_{out}=\mathbf{H}\,\mathbf{f}$, where $\mathbf{H}=h(\mathcal{L})\equiv\sum_{i=1}^{N}h(\lambda_i)\mathbf{u}_i\mathbf{u}_i^T=\mathbf{U}\,\mathrm{diag}\{h(\lambda_i)\}\,\mathbf{U}^T$ can be considered as a transformation matrix. Graph filtering therefore in general requires the knowledge of the eigenvalues/eigenvectors of the underlying graph G. An eigendecomposition of a large graph (with large N) is however computationally expensive. However when the spectral filter is given by $h(\lambda)=\sum_{k=0}^{K}b_k\lambda^k$ (K degree polynomial with coefficients b_k), it can be readily shown (using the identity $\mathcal{L}^k=\mathbf{U}\,\mathrm{diag}\{\lambda_i^k\}\,\mathbf{U}^T$) that the transformation matrix is given by $\mathbf{H}=\sum_{k=0}^{K}b_k\mathcal{L}^k$. Explicit knowledge of the eigenvalues/eigenvectors is therefore not required for filtering. Only powers of the Laplacian is required which is much less expensive computationally. Another important property of polynomial filters is localization. A K-hop (K integer) local neighborhood for vertex i, denoted by $\mathcal{N}(i, K)$, is defined as the set of (other) vertices that are connected to vertex i by no more than K edges. For example in Fig. 1, $\mathcal{N}(3, 1)=\{1, 2, 6\}$ and $\mathcal{N}(6, 2)=\{1, 2, 3, 4, 5, 7\}$. A filter $h(\mathcal{L})$ is *K-hop localized* if the output $f_{out}(i)$ is determined *only* by input values $f(j)$ in $\mathcal{N}(i, K)$. It can be shown that a degree K