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Discrete estimators of characteristics for periodically correlated time series

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ABSTRACT

Results of an investigation of the characteristic estimator properties for periodically correlated time series obtained on the basis of finite data length are given. The formulae for the bias and variance of the estimators for mean and covariance function Fourier coefficients are found. The conditions for the choice of sampling interval value, for which aliasing effects do not appear, are obtained. The interpolation formulae for the mean and covariance function estimates are derived. The dependencies of the statistical characteristics of the estimators on sampling interval and sample size for modulated signals are analyzed.

1. Introduction

Signal models in the form of periodically correlated random processes (PCRP) are widely used in different areas of science and technology [1–10]. The first and second order moment functions of this class of non-stationary random processes adequately describe the properties of signal recurrence and stochasticity. These models combine and develop both the deterministic and stochastic approaches. The first is based on models in the form of periodic functions, and the second is based on models in the form of stationary random processes. PCRP, as individual cases, contain polyharmonic, additive, multiplicative, additive–multiplicative, quadrature and other models for the description of the interaction between recurrence and stochasticity [11]. The capability of joint interpretation of different types of stochastic recurrence on the basis of PCRP is represented by [12–14]:

$$\xi(t)=\sum_{k\in Z}\xi_k(t)e^{ik\omega_0 t}, \tag{1}$$

where $\xi_k(t)$ are jointly stationary processes, $\omega_0=2\pi/P$, P is the period, and Z is the set of integer numbers. If PCRP is a random process with a finite average power, that is $\frac{1}{P}\int_0^P E|\xi(t)|^2dt<\infty$, then series (1) is convergent in sense [1,2,14]:

$$E\left\{\frac{1}{P}\int\limits_0^P\left|\xi(t)-\sum_{k=-L}^{L}\xi_k(t)e^{ik\omega_0 t}\right|^2dt\right\}\xrightarrow{L\to\infty}0.$$

It follows from expression (1) that PCRP in general is a superposition of the amplitude and phase modulated harmonics with frequencies multiple to the main ω_0. The mean function, covariance function, spectral density and their Fourier coefficients describe properties of such a modulation [2,9].

For investigation of the probabilistic structure of the stochastic recurrence of physical phenomena on the basis of experimental data, methods for PCRP statistical analysis are used. Among them are: coherent [2,5,19] and component [2,6,17,18] methods, the least squares method [19], methods of linear combing [20], and band [21] filtration. Each of these methods has its own features and may be used depending on the concrete situation. Sampling is a required procedure of signal statistical analysis when technical tools are used. The choice of sampling interval is the most important issue for its implementation. Traditionally, the sampling interval is chosen in accordance to the Kotel'nikov–Shannon theorem, where the signal's highest frequency is known [22,23]. The lowest frequency of sampling is named the Nyquist frequency. For the case of the Nyquist sampling rate, the exact interpolation formula (for band-limited signals) can be obtained [23]. However, using the Nyquist sampling rate for statistical processing of experimental data may lead to the appearance of significant errors in the results. The sampling interval value in this case should be chosen on the results of the analysis of statistical properties of probabilistic characteristic estimators. The assumption for the choice can only be given using solutions to concrete statistical tasks. This

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especially concerns statistics of non-stationary random processes, where even in second-order theory we deal with the covariance functions $b(t,u) = E\overset{\circ}{\xi}(t)\overset{\circ}{\xi}(t+u)$, $\overset{\circ}{\xi}(t) = \xi(t) - m(t)$, $m(t) = E\xi(t)$, which depend on two variables: time t and lag u; the properties of their behavior on each of the variables may be very different. The covariance function of PCRP $b(t,u)$, for example, periodically varies in time $b(t+P,u) = b(t,u)$, and, as a rule, damps as the lag u increases. The damping character and its rapidity for different physical phenomena may differ significantly [2,6,9]. It causes bi-frequency spectral density features – two-dimensional Fourier transformation of covariance function. It is discrete relative to so-called cyclic frequencies, whose set is defined by the type of time periodicity of $b(t,u)$, and is continuous relative to the frequencies, which represent the dependence of $b(t,u)$ on the lag.

The discrete estimators of Fourier coefficients of the mean and covariance functions for PC time series are analyzed in this paper. Mean and covariance functions are estimated using coherent methods [2,5,7,16], when the period of non-stationarity P is known. This is a continuation of the work by Javors'kyj [24], in which the conditions of mean square convergence of estimators were obtained. Note that discrete estimators of PCRP probabilistic characteristics have been considered in the literature [3–5,7,14,25–27]. The cycloergodic properties of non-stationary discrete parameter processes were investigated in [25,26]. The link between continuous and discrete cases for cyclic second-order statistics is analyzed in [3–5,27]. It was shown that for time-sampled data in general, the aliasing phenomenon occurred. The limit of bi-frequency spectrum, for example, in the discrete case is the sum of its values for continuous time; such values are shifted relative to both frequencies towards multiple values of the sample frequency. The aliasing effect can be avoided when PC time series are band limited. The sample frequency that ensures the absence of an aliasing effect in the area of cyclic frequencies should be twice as great as the Nyquist frequency. The Nyquist sampling rate ensures the absence of the aliasing effect in the area of cyclic frequencies. Similar comparative analysis of continuous and discrete-time statistics inclusive of higher-order statistics has been provided for near-PC time series and their generalizations [28–32]. We should note that they characterize only the asymptotical properties of the respective statistics. Stochastic components of the discrete estimation errors that are inherent for finite time length remained beyond the authors' attention. Such components can be carried out using a statistical approach based on analysis of moment functions of the respective characteristic estimators. On the basis of the last count, the systematic and root square errors of discrete analysis can be calculated. These errors should be used for efficiency comparison of continuous and discrete estimation. One must ensure that conditions of aliasing absence mentioned above are obtained using asymptotical expressions for systematic errors of respective characteristics estimations (this issue will be discussed further later on).

We should note that analyzing the deterministic and stochastic parts of experimental time series properties, i.e. estimators of mean and covariance function and also their Fourier coefficients, are, as a rule, investigated separately. The effects caused by sampling appear in the estimators of these characteristics differently, which is why they should be analyzed separately.

Using the coherent method [2,16] the estimators of mean function and covariance function for some moments of time are defined on the basis of averaging of the signal values taken through the period P. On the basis of these estimators, Fourier components estimators are built. The task of sampling in this case should be considered from two positions. First, the interval between moments of time, for which values of mean function and covariance function are estimated, should be grounded. Taking into account the time recurrence of these characteristics, it is concluded that trigonometric interpolation [33] should be used for solving this task. The second task to be solved is the grounding of the sampling interval value by replacement of the respective integral transformations by integral sums when the Fourier components of the mean function and covariance function are calculated. In Section 2, the discrete estimators of mean function are analyzed. In Section 3, we investigate the Fourier components of the covariance function, the so-called covariance components [1,2,34,35]. The estimator properties in the article are analyzed under assumption that realization length is $T = NP$, where N – natural number. The cyclic leakage [3,10] does not appear in this case [18]. In Section 4, the obtained results are specified for amplitude- and phase-modulated signals, one of the simplest models of PCRP. The numeric results from the analysis allow one to detail the properties of discrete estimators.

2. Estimation of mean function Fourier coefficients

The mean function $m(t) = E\xi(t)$ and covariance function $b(t,u) = E\overset{\circ}{\xi}(t)\overset{\circ}{\xi}(t+u)$ of PCRP are periodic functions of time: $m(t) = m(t+P)$, $b(t,u) = b(t+P,u)$. If the conditions

$$\int_0^P |m(t)|\mathrm{d}t < \infty, \qquad \int_0^P |b(t,u)|\mathrm{d}t < \infty$$

are satisfied, they can be represented in Fourier series form

$$m(t) = \sum_{k\in Z} m_k \mathrm{e}^{ik\omega_0 t} = m_0 + \sum_{k\in N}\left(m_k^c \cos k\omega_0 t + m_k^s \sin k\omega_0 t\right),$$

$$b(t,u) = \sum_{k\in Z} B_k(u)\mathrm{e}^{ik\omega_0 t}$$

$$= B_0(u) + \sum_{k\in N}\left[B_k^c(u)\cos k\omega_0 t + B_k^s(u)\sin k\omega_0 t\right],$$

where $m_k = \frac{1}{2}(m_k^c - i m_k^s)$, $B_k(u) = \frac{1}{2}[B_k^c(u) - i B_k^s(u)]$, and N is a set of natural numbers.

Note that for functions $m(t)$ and $b(t,u)$ the Parseval's relations are inherent:

$$\frac{1}{P}\int_0^P |m(t)|^2 dt = \sum_{k\in Z}|m_k|^2, \qquad \frac{1}{P}\int_0^P |b(t,u)|^2 dt = \sum_{k\in Z}|B_k(u)|^2.$$

It follows from these equalities that the infinite sums of the square absolute values of the Fourier coefficients m_k and $B_k(u)$ in the case of the absolute integrability of $m(t)$ and $b(t,u)$ are always finite. We should reiterate that $|m_k| \to 0$ and $|B_k(u)| \to 0$ at $k \to \infty$.

Placing $P = (M+1)h$, h is the sampling interval, M is a natural number and the realization length is $T = NP$. The coherent estimator of the mean function [2,16] at the points $t_n = nh$, $n = \overline{0, M+1}$, has the form:

$$\hat{m}(nh) = \frac{1}{N}\sum_{k=0}^{N-1}\xi\left[\left(n + k(M+1)\right)h\right]. \tag{2}$$

Fourier coefficients estimators can be found on this basis:

$$\hat{m}_0 = \frac{1}{M+1}\sum_{n=0}^{M}\hat{m}(nh), \tag{3}$$

$$\hat{m}_l^c = \frac{2}{M+1}\sum_{n=0}^{M}\hat{m}(nh)\cos l\frac{2\pi}{M+1}n, \tag{4}$$

$$\hat{m}_l^s = \frac{2}{M+1}\sum_{n=0}^{M}\hat{m}(nh)\sin l\frac{2\pi}{M+1}n. \tag{5}$$