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Birth and death in discrete Morse theory

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ABSTRACT

Suppose M is a finite cell decomposition of a space X and that for $0 = t_0 < t_1 < \cdots < t_r = 1$ we have a discrete Morse function $F_{t_i} : M \to \mathbf{R}$. In this paper, we study the births and deaths of critical cells for the functions F_{t_i} and present an algorithm for pairing the cells that occur in adjacent slices. We first study the case where the cell decomposition of X is the same for each t_i, and then generalize to the case where they may differ. This has potential applications in topological data analysis, where one has function values at a sample of points in some region in space at several different times or at different levels in an object.

1. Introduction

The purpose of this paper is to study the discrete analogue of the following phenomenon in classical smooth Morse theory. Suppose that N is a smooth manifold and that we have a family of functions $f : N \times I \to \mathbf{R}$ such that the various $f_t : N \to \mathbf{R}$ are generically Morse; that is, for almost all t, the function f_t has only nondegenerate critical points. Then as t varies, the critical points of the f_t move around in N. Sometimes a critical point is "born"; that is, a new critical point appears at some time t_0. At other times, critical points "die". Generically, critical points are born and die in pairs. Such

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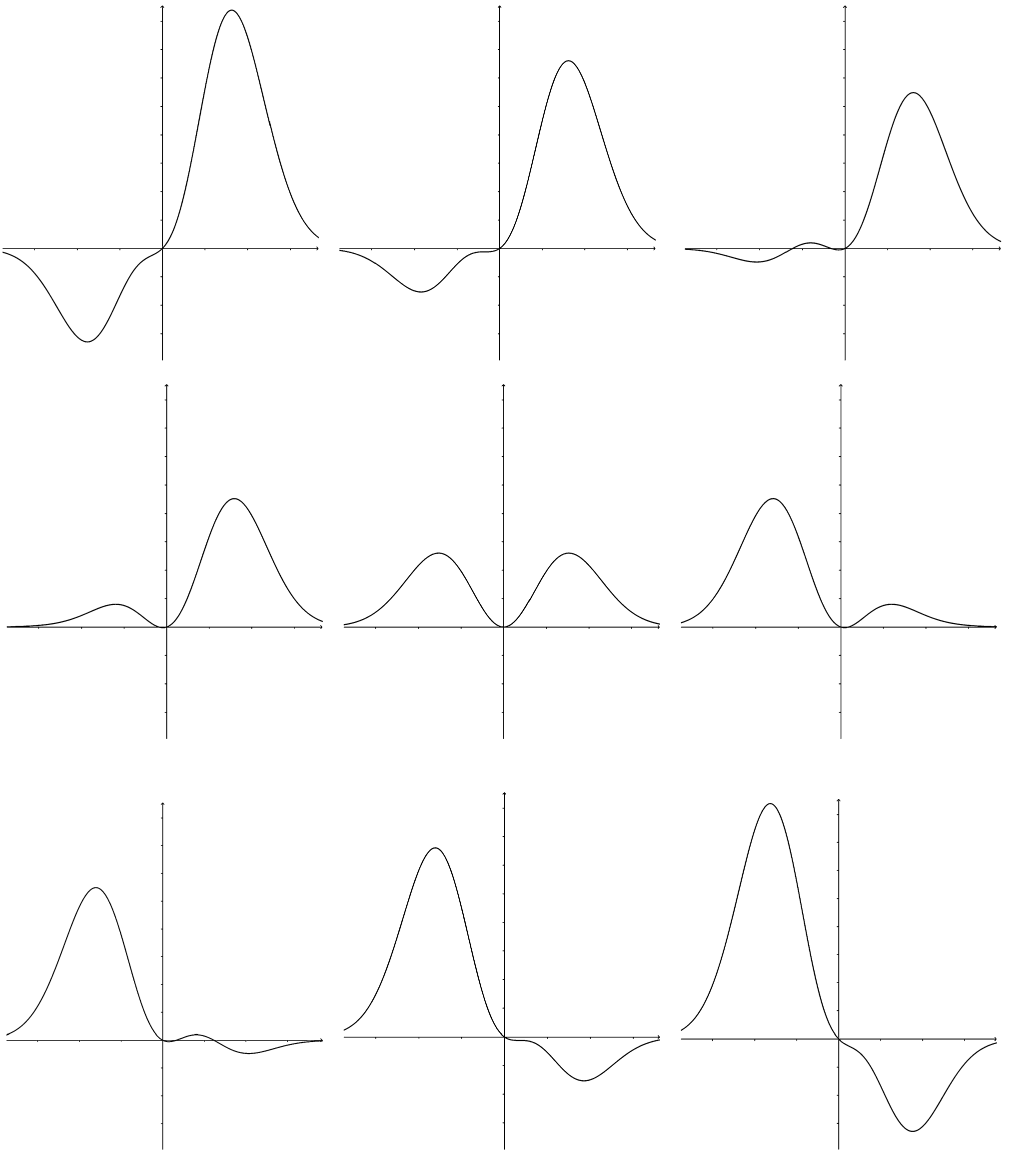

Fig. 1. The curves $f(x,t)$ for $t = -3, -2.08, -1.5, -1, 0, 1, 1.5, 2.08, 3$.

events are isolated since the critical points of a Morse function are separated; we call such points in $N \times I$ *birth–death points*.

Consider the following simple example. Let $N = \mathbf{R}$ and consider the family $f : N \times I \to \mathbf{R}$ defined by

$$f(x,t) = e^{-x^2/2}\left(\frac{x^4}{2} - 3tx^3 + 6x^2 - tx\right).$$

Each f_t is a Morse function except for a pair of values $t \approx \pm 2.08$. Fig. 1 shows the graphs of $f(x,t)$ for a few values of t. Note the evolution of the critical points as we pass through various t values. In intermediate stages, we see the appearance of degenerate critical points ($t \approx -2.08$ and $t \approx 2.08$).