ARTICLE IN PRESS

Available online at www.sciencedirect.com

ScienceDirect

Expo. Math. ▮ (▮▮▮▮) ▮▮▮–▮▮▮

EXPOSITIONES MATHEMATICAE

www.elsevier.com/locate/exmath

On a dynamical local–global principle in Mordell–Weil type groups

Stefan Barańczuk

Faculty of Mathematics and Computer Science, Adam Mickiewicz University, ul. Umultowska 87, Poznań, Poland

Received 4 December 2015; received in revised form 6 June 2016

Abstract

In this paper we consider certain dynamical local–global principle for Mordell–Weil type groups over number fields like S-units, abelian varieties with trivial ring of endomorphisms and odd algebraic K-theory groups.

MSC 2010: primary 37P55; secondary 11G10; 11R70

Keywords: Reduction maps; Abelian varieties; K-theory groups

1. Introduction

The mathematical theory of dynamical systems studies, given a set S and a map $\phi: S \to S$ mapping the set S to itself, the behaviour of the iterates of the map ϕ,

$$\phi^n = \underbrace{\phi \circ \ldots \circ \phi}_{n \text{ times}},$$

in particular, for given $s \in S$, the behaviour of the orbit of s, i.e. the set

$$\mathcal{O}_\phi(s) = \left\{\phi^n(s) : n \geq 0\right\}.$$

If S is equipped with a group structure then we might consider only those ϕ's that are endomorphisms. In this note we investigate the following groups of number-theoretic interest:

E-mail address: stefbar@amu.edu.pl.

http://dx.doi.org/10.1016/j.exmath.2016.07.001
0723-0869/

Please cite this article in press as: S. Barańczuk, On a dynamical local–global principle in Mordell–Weil type groups, Expo. Math. (2016), http://dx.doi.org/10.1016/j.exmath.2016.07.001

1. $R_{F,S}^{\times}$, S-units groups, where F is a number field and S is a finite set of ideals in the ring of integers R_F,
2. $A(F)$, Mordell–Weil groups of abelian varieties over number fields F with $\mathrm{End}_{\bar{F}}(A) = \mathbb{Z}$,
3. $K_{2n+1}(F)$, $n > 0$, odd algebraic K-theory groups.

To deal with them it is enough for us to introduce the following abstract nonsense axiomatic setup:

Let B be an abelian group and $r_v \colon B \to B_v$ be an infinite family of groups homomorphisms whose targets B_v are finite abelian groups. We will use the following notation:

$P \bmod v$	denotes $r_v(P)$ for $P \in B$
$P = Q \bmod v$	means $r_v(P) = r_v(Q)$ for $P, Q \in B$
Λ_{tors}	the torsion part of a subgroup $\Lambda < B$
$\mathrm{ord}\, T$	the order of a torsion point $T \in B$
$\mathrm{ord}_v P$	the order of a point $P \bmod v$
$l^k \parallel n$	means that l^k exactly divides n, i.e. $l^k \mid n$ and $l^{k+1} \nmid n$ where l is a prime number, k a nonnegative integer and n a natural number.

We impose the following two assumptions on the family $r_v \colon B \to B_v$:

Assumptions. 1. Let l be a prime number, $(k_1, \dots, k_m)$ a sequence of nonnegative integers. If $P_1, \dots, P_m \in B$ are points linearly independent over $\mathbb{Z}$, then there is an infinite family of primes v in F such that $l^{k_i} \parallel \mathrm{ord}_v P_i$ if $k_i > 0$ and $l \nmid \mathrm{ord}_v P_i$ if $k_i = 0$.
2. For almost all v the map $B_{\mathrm{tors}} \to B_v$ is injective.

(For the families of groups we described above the axioms are valid, in particular Assumption 1 is fulfilled by [5, Theorem 5.1], and Assumption 2 by [2, Lemma 3.11]. The validity of the technical hypothesis[1] from the first sentence of Theorem 5.1 in [5] is clear in the K-theory groups case (resp. $R_{F,S}^{\times}$ case) since here ρ_l is simply one-dimensional representation given by the $(n+1)$th tensor power of cyclotomic character (resp. by cyclotomic character); in abelian varieties case it is asserted by Corollary 1 in [8].)

In this short note we prove the following theorem, inspired by Joseph Silverman's question stated in his private communications with Grzegorz Banaszak:

Theorem. *Let Λ be a subgroup of B, $P \in B$ be a point of infinite order and ϕ be a natural number. Then the following are equivalent:*

- *For almost every v*

$$\mathcal{O}_{\phi}\,(P \bmod v) \cap \Lambda \bmod v \neq \emptyset. \tag{1}$$

- $\mathcal{O}_{\phi}\,(P) \cap \Lambda \neq \emptyset.$

where $\phi^k P$ means multiplying P by integer ϕ^k; notice that the maps given by multiplication by integers are endomorphisms, thus indeed we deal here with iterates of endomorphisms mentioned at the beginning of the introduction. For $\phi = 1$ the statement of the

[1] Note the misprint in its formulation: $\rho(G_F)$ should read $\rho_l(G_F)$.

Please cite this article in press as: S. Barańczuk, On a dynamical local–global principle in Mordell–Weil type groups, Expo. Math. (2016), http://dx.doi.org/10.1016/j.exmath.2016.07.001