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VARIATIONS ON A THEME BY HIGMAN

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ABSTRACT. We propose elementary and explicit presentations of groups that have no amenable quotients and yet are SQ-universal. Examples include groups with a finite $K(\pi, 1)$, no Kazhdan subgroups and no Haagerup quotients.

1. INTRODUCTION

In 1951, G. Higman defined the group

$$\mathrm{Hig}_n = \langle a_i \ (i \in \mathbf{Z}/n\mathbf{Z}) : [a_{i-1}, a_i] = a_i \rangle \tag{1}$$

and proved that for $n \geq 4$ it is infinite without non-trivial finite quotient [9]. Since the presentation (1) is explicit and simple, A. Thom suggested that Hig_n is a good candidate to contradict approximation properties for groups and proved such a result in [21]. Perhaps the most elusive approximation property is still *soficity* [7, 22]; but a non-sofic group would in particular not be residually *amenable*, a statement we do not know for the Higman groups (cf. also [8]). The purpose of this note is to propound variations of Higman's construction with no non-trivial amenable quotients at all.

There are several known sources of groups without amenable quotients since it suffices to take a (non-amenable) *simple* group to avoid all possible quotients. However, as Thelonius Sphere Monk observed, *simple ain't easy*. To wit, one had to wait until the break-through of Burger–Mozes [2, 3] for simple groups of *type F*, i.e. admitting a finite $K(\pi, 1)$. Before this, no torsion-free finitely presented simple groups were known.

The examples below are of a completely opposite nature because they admit a wealth of quotients: indeed, like Hig_n, they are *SQ-universal*, i.e. contain any countable group in a suitable quotient. It follows that they have uncountably many quotients [14, §III], despite having no amenable quotients.

We shall start with the easiest examples, whose cyclic structure is directly inspired by (1). Below that, we propose a cleaner construction, starting from copies of $\mathbf{Z}$ only, which might be a better candidate to contradict approximation properties; the price to pay is to replace the cycle by a more complicated graph.

Disclaimer. No claim is made to produce the first examples of groups with a hodgepodge of sundry properties (for instance, if G is a Burger–Mozes group, then $G * G$ satisfies many properties of G_n in Theorem 2 below, though with "amenable" instead of "Haagerup").
Our goal is to suggest transparent presentations for which the stated properties are explicit and their proofs effective.