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Minimum distance and the minimum weight codewords of Schubert codes

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ABSTRACT

We consider linear codes associated to Schubert varieties in Grassmannians. A formula for the minimum distance of these codes was conjectured in 2000 and after having been established in various special cases, it was proved in 2008 by Xiang. We give an alternative proof of this formula. Further, we propose a characterization of the minimum weight codewords of Schubert codes by introducing the notion of Schubert decomposable elements of certain exterior powers. It is shown that codewords corresponding to Schubert decomposable elements are of minimum weight and also that the converse is true in many cases. A lower bound, and in some cases, an exact formula, for the number of minimum weight codewords of Schubert codes is also given. From a geometric point of view, these results correspond to determining the maximum number of $\mathbb{F}_q$-rational points that can lie on a hyperplane section of a Schubert variety in a Grassmannian with its nondegenerate embedding in a projective subspace of the Plücker projective space, and also the number of hyperplanes for which the maximum is attained.

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1. Introduction

Fix a prime power q and positive integers ℓ, m with $\ell \leq m$. Let $\mathbb{F}_q$ denote the finite field with q elements and let V be a vector space over $\mathbb{F}_q$ of dimension m. To the Grassmannian $G_{\ell,m}$ of all ℓ-dimensional linear subspaces of V, one can associate in a natural way an $[n,k]_q$-code, i.e., a q-ary linear code of length n and dimension k, where

$$n = \begin{bmatrix} m \\ \ell \end{bmatrix}_q := \frac{(q^m-1)(q^m-q)\cdots(q^m-q^{\ell-1})}{(q^\ell-1)(q^\ell-q)\cdots(q^\ell-q^{\ell-1})} \quad \text{and} \quad k = \binom{m}{\ell}. \tag{1}$$

This code is denoted by $C(\ell, m)$ and is called *Grassmann code*. The study of Grassmann codes goes back to the work of Ryan [15,16,18] in the late 1980's and was continued by Nogin [13] and several authors (see, e.g., [6,7,10,5,2] and the references therein). It is now known that Grassmann codes possess a number of interesting properties. For instance, their minimum weight is known and is given by the following beautiful formula of Nogin [13]:

$$d\left(C(\ell, m)\right) = q^\delta \quad \text{where} \quad \delta := \ell(m-\ell). \tag{2}$$

Furthermore, several generalized Hamming weights are known, the automorphism group has been determined and is known to be fairly large, the duals of Grassmann codes have a very low minimum distance (namely, 3) and the minimum weight codewords of $C(\ell,m)^\perp$ generate $C(\ell,m)^\perp$. In fact, as the results of [2,14] show, Grassmann codes can be regarded as regular LDPC codes and also as a Tanner codes with a small component code, namely, $C(1,2)$.

Schubert codes are a natural generalization of Grassmann codes and were introduced in [6] around the turn of the last century. These are linear codes $C_\alpha(\ell,m)$ associated to Schubert subvarieties $\Omega_\alpha(\ell,m)$ of the Grassmannian $G_{\ell,m}$ and are indexed by ℓ-tuples $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ of positive integers with $1 \leq \alpha_1 < \cdots < \alpha_\ell \leq m$. The Grassmann codes are a special case where $\alpha_i = m - \ell + i$ for $i = 1, \ldots, \ell$. It was shown in [6] that the minimum distance of $C_\alpha(\ell,m)$ satisfies the inequality

$$d\left(C_\alpha(\ell,m)\right) \leq q^{\delta(\alpha)} \quad \text{where} \quad \delta(\alpha) := \sum_{i=1}^{\ell}(\alpha_i - i). \tag{3}$$

Further, it was conjectured in [6] that the inequality in (3) is, in fact, an equality. We will refer to this conjecture as the Minimum Distance Conjecture, or in short, the MDC. When $\alpha_i = m - \ell + i$ for $i = 1, \ldots, \ell$, we have $\delta(\alpha) = \delta$ and so the MDC holds, thanks to (2). In the case $\ell = 2$, the MDC was proved in the affirmative by Chen [3] and, independently, by Guerra and Vincenti [9]. An explicit formula for the length n_α and dimension k_α of $C_\alpha(\ell,m)$ in the case $\ell = 2$ was also given in [3], while [9] gave a general, even if complicated, formula for n_α for arbitrary ℓ. Later, in [8], the MDC was established