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Strongly verbally closed groups

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A B S T R A C T

It was recently proven that all free and many virtually free verbally closed subgroups are algebraically closed in *any* group. We establish sufficient conditions for a group that is an extension of a free non-abelian group by a group satisfying a non-trivial law to be algebraically closed in any group in which it is verbally closed. We apply these conditions to prove that the fundamental groups of all closed surfaces, except the Klein bottle, and almost all free products of groups satisfying a non-trivial law are algebraically closed in any group in which they are verbally closed.

1. Introduction

A subgroup H of a group G is called *verbally closed* (in G) [8] (see also [10], [11], [4], [7]) if any equation of the form $w(x_1, \ldots, x_n) = h$, where $w(x_1, \ldots, x_n) \in F_n(x_1, \ldots, x_n)$ and $h \in H$, having a solution in G has a solution in H too.

A subgroup H of a group G is called *algebraically closed* (in G) if any system of equations of the form $\{w_1(x_1, \ldots, x_n, H) = 1, \ldots, w_m(x_1, \ldots, x_n, H) = 1\}$, where $w_i(x_1, \ldots, x_n, H) \in F_n(x_1, \ldots, x_n) * H$, having a solution in G has a solution in H too.

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A subgroup H of a group G is called *retract* (of G) if G is a semidirect product of a normal subgroup N and H (i.e. $G = N \rtimes H$).

It is easy to see that any retract is an algebraically closed subgroup and any algebraically closed subgroup is a verbally closed subgroup. Thus, the question naturally arises: under what conditions on a subgroup H and a group G the revers implications hold? It is known (see, e.g., [4]) that both of these reverse implications do not hold in general. The following was established in [8]

R1) *if G is finitely presented and H is a finitely generated algebraically closed subgroup in G, then H is a retract of G;*
R2) *if G is finitely generated over H*[1] *and H is a equationally Noetherian*[2] *algebraically closed subgroup in G, then H is a retract of G.*

For the class of verbally closed subgroups, no similar structural descriptions in known. However, in finitely generated free groups (see [8]) and finitely generated free nilpotent groups (see [11]) the situation is rather simple: verbally closed subgroups, algebraically closed subgroups and retracts are the same things. In article [4], the following theorem was established

Theorem [4]. *Let G be any group and let H be its verbally closed virtually free infinite non-dihedral*[3] *subgroup containing no infinite abelian noncyclic subgroups. Then*

1) *H is algebraically closed in G;*
2) *if G is finitely generated over H, then H is a retract of G.*

We call H a *strongly verbally closed group* if H is an algebraically closed subgroup in any group containing H as a verbally closed subgroup. Notice, that the first assertion of the given above theorem describes a certain class of strongly verbally closed groups (in particular, all non-trivial free groups belong to this class). Abelian groups is another class of strongly verbally closed groups (see Corollary 4). In this article we establish sufficient conditions for a group H that is an extension of a free non-abelian group by a group satisfying a non-trivial law[4] to be strongly verbally closed. These conditions can be used to establish strong verbal closedness of rather wide class of groups. For instance, is Section 2 we apply them to prove the following

[1] A group G is *finitely generated over H* if $G = \langle H, X \rangle$ for some finite subset $X \subseteq G$.
[2] A group H is *equationally Noetherian* if any system of equations with coefficients from H and finitely many unknowns is equivalent to its finite subsystem.
[3] For an infinite group, *non-dihedral* means non-isomorphic to the free product of two groups of order two.
[4] Let $I(x_1, \ldots, x_r)$ be an element of the free group $F_r(x_1, \ldots, x_r)$ with a basis $x_1, \ldots, x_r$, then we say that a *group G satisfies the law I* if $I(g_1, \ldots, g_r) = 1$ in G for all $g_1, \ldots, g_r \in G$.