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Normal Sally modules of rank one

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In this paper, we explore the structure of the normal Sally modules of rank one with respect to an $\mathfrak{m}$-primary ideal in a Nagata reduced local ring R which is not necessary Cohen–Macaulay. As an application of this result, when the base ring is Cohen–Macaulay analytically unramified, the extremal bound on the first normal Hilbert coefficient leads to the depth of the associated graded rings $\overline{\mathcal{G}}$ with respect to a normal filtration is at least $\dim R-1$ and $\overline{\mathcal{G}}$ turns in to Cohen–Macaulay when the third normal Hilbert coefficient is vanished.

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1. Introduction

Throughout this paper, let R be an analytically unramified Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim R > 0$. Let I be an $\mathfrak{m}$-primary ideal of R and suppose that I contains a parameter ideal $Q = (a_1, a_2, ..., a_d)$ of R as a reduction. Let $\ell_R(M)$ denote the length of an R-module M and $\overline{I^{n+1}}$ denote the integral closure of I^{n+1} for each $n \geq 0$. Since R is an analytically unramified, there are integers $\{\overline{\mathrm{e}_i}(I)\}_{0\leq i\leq d}$ such that the equality

$$\ell_R(R/\overline{I^{n+1}}) = \overline{\mathrm{e}_0}(I)\binom{n+d}{d} - \overline{\mathrm{e}_1}(I)\binom{n+d-1}{d-1} + ... + (-1)^d\overline{\mathrm{e}_d}(I)$$

holds true for all integers $n \gg 0$, which we call the normal Hilbert coefficients of R with respect to I. We will denote by $\{\mathrm{e}_i(I)\}_{0\leq i\leq d}$ the ordinary Hilbert coefficients of R with respect to I. Let

$$\mathcal{R} = \mathrm{R}(I) := R[It] \text{ and } T = \mathrm{R}(Q) := R[Qt] \subseteq R[t]$$

denote, respectively, the Rees algebra of I and Q, where t stands for an indeterminate over R. Let

$$\mathcal{R}' = \mathrm{R}'(I) := R[It, t^{-1}] \text{ and } \mathcal{G} = \mathcal{G}(I) := \mathcal{R}'/t^{-1}\mathcal{R}' \cong \oplus_{n\geq 0}I^n/I^{n+1}$$

denote, respectively, the extended Rees algebra of I and the associated graded ring of R with respect to I. Let $\overline{\mathcal{R}}$ denote the integral closure of $\mathcal{R}$ in $R[t]$ and $\overline{\mathcal{G}} = \oplus_{n\geq 0}\overline{I^n}/\overline{I^{n+1}}$ denote the associated graded ring of the normal filtration $\{\overline{I^n}\}_{n\in\mathbb{Z}}$. Then $\overline{\mathcal{R}} = \oplus_{n\geq 0}\overline{I^n}t^n$ and $\overline{\mathcal{R}}$ is a module-finite extension of $\mathcal{R}$ since R is analytically unramified (see [14, Corollary 9.2.1]). For the reduction Q of I, the reduction number of $\{\overline{I^n}\}_{n\in\mathbb{Z}}$ with respect to Q is defined by

$$r_Q(\{\overline{I^n}\}_{n\in\mathbb{Z}}) = \min\{r \in \mathbb{Z} \mid \overline{I^{n+1}} = Q\overline{I^n}, \text{for all } n \geq r\}.$$

The notion of Sally modules of normal filtrations was introduced by [1] in order to find the relationship between a bound on the first normal Hilbert coefficients $\overline{\mathrm{e}_1}(I)$ and the depth of $\overline{\mathcal{G}}$ when R in an analytically unramified Cohen–Macaulay rings R. Following [1], we generalize the definition of normal Sally modules to the non-Cohen–Macaulay cases, and we define the normal Sally modules $\overline{S} = \overline{S}_Q(I)$ of I with respect to a minimal reduction Q to be the cokernel of the following exact sequence

$$0 \longrightarrow \overline{I}T \longrightarrow \overline{\mathcal{R}}_+(1) \longrightarrow \overline{S} \longrightarrow 0$$

of graded T-modules. Since $\overline{\mathcal{R}}$ is a finitely generated T-module, so is $\overline{S}$ and we get

$$\overline{S} = \oplus_{n\geq 1}\overline{I^{n+1}}/Q^n\overline{I}$$

by the following isomorphism