Journal of Algebra 489 (2017) 138–164

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

The transpose of modules relative to subcategories ☆

Yanbo Zhou, Xiaosheng Zhu *

Department of Mathematics, Nanjing University, Nanjing 210093, PR China

A R T I C L E   I N F O

Article history:
Received 6 November 2016
Available online 4 July 2017
Communicated by Louis Rowen

MSC:
18G25
16E05

Keywords:
Transpose
Subcategories
Semidualizing bimodules
Ext-injective generators
Syzygies

A B S T R A C T

Let R be a left noetherian ring and S a right noetherian ring, and let $\mathcal{X}$ be a subcategory of finitely generated left R-modules and ${}_RB_S$ a finite (R,S)-bimodule. As a generalization of the Auslander transpose, replacing a projective presentation by an $\mathcal{X}$-presentation and the functor $\mathrm{Hom}_R(-,R)$ by the functor $\mathrm{Hom}_R(-,B)$, we introduce the $\mathcal{X}(B)$-transpose of an R-module M admitting an $\mathcal{X}$-presentation. For a suitable subcategory $\mathcal{X}$, some useful properties of $\mathcal{X}(B)$-transposes are obtained. It is shown that the $\mathcal{X}(B)$-transpose shares many nice properties with the Auslander transpose. Some known results are obtained as corollaries.

0. Introduction

In this paper, R is a left noetherian ring and S is a right noetherian ring. We use mod-R (mod-S^{op}) to denote the category of finitely generated left R-modules (right S-modules). Let $Q \in$ mod-R. Denote add$Q$ the full subcategory of mod-R whose objects

☆ Supported by the National Natural Science Foundation of China (No. 11571165).

* Corresponding author.
E-mail address: zhuxs@nju.edu.cn (X. Zhu).

http://dx.doi.org/10.1016/j.jalgebra.2017.06.024

are the direct summands of finite direct sum copies of Q. All modules considered are finitely generated.

The transpose was originally introduced by Auslander and Bridger in [2]. It plays an important role in the construction of the well-known Auslander–Reiten sequence (see [1]). Martsinkovsky and Strooker introduced the notion of linkage for modules over non-commutative semiperfect rings by using the Auslander–Bridger transpose in [13]. In order to study the theory of linkage for modules of finite G_C-dimension, replacing the functor $\mathrm{Hom}_R(-,R)$ by the functor $\mathrm{Hom}_R(-,C)$, Dibaei and Sadeghi introduced in [5] the C-transpose $\mathrm{Tr}_C M$ for an R-module M with respect to a semidualizing R-module C. Substituting the Gorenstein projective presentation for the projective presentation, Huang and Huang introduced the Gorenstein transpose in [12]. An R-module M is called *C-projective* if there is a projective R-module P such that $M = C \otimes_R P$. In [15], replacing the projective presentation by the C-projective presentation and the functor $\mathrm{Hom}_R(-,R)$ by the functor $\mathrm{Hom}_R(-,C)$ for a semidualizing R-module C, Salimi, Tavasoli, Moradifar and Yassemi introduced the relative Auslander dual (see [15, Definition 3.4]). Let $\mathcal{X}$ be a subcategory of mod-R. By an *$\mathcal{X}$-resolution* of an R-module M, we mean an exact sequence $\cdots \to X_2 \to X_1 \to X_0 \to M \to 0$ with $X_i \in \mathcal{X}$ for all $i \geq 0$. For an R-module M admitting an $\mathcal{X}$-resolution, in this paper, we introduce a more general relative transpose for M by replacing the projective presentation by the $\mathcal{X}$-presentation and the functor $\mathrm{Hom}_R(-,R)$ by the functor $\mathrm{Hom}_R(-,B)$ with respect to an R-module B. This relative transpose for M is called *$\mathcal{X}(B)$-transpose* of M and denoted by $\mathrm{Tr}_{\mathcal{X}}^B M$. The main purpose of the present paper is to generalize some well-known results about the Auslander–Bridger transpose to the setting of this relative transpose. The structure of the paper is as follows.

In Section 1, we give the definition of the relative transpose with respect to a suitable subcategory $\mathcal{X}$ and a finite (R,S)-module ${}_RB_S$ and consider some interesting examples.

An $\mathcal{X}$-presentation of M is called *proper* if it is $\mathrm{Hom}_R(X,-)$-exact for all $X \in \mathcal{X}$. An $\mathcal{X}(B)$-transpose for M obtained by a proper $\mathcal{X}$-presentation is called a *proper $\mathcal{X}(B)$-transpose*. Two modules M and N are said to be *$\mathcal{X}^B$-equivalent* if $M \oplus X^B \cong N \oplus Y^B$ with $X,\ Y \in \mathcal{X}$ and $(-)^B := \mathrm{Hom}_R(-,B)$. In Section 2, we prove that two proper $\mathcal{X}(B)$-transposes for M are $\mathcal{X}^B$-equivalent when $\mathcal{X}$ is a subcategory of mod-R closed under extensions and either direct summand or kernels of epimorphisms (see Theorem 2.3). Some results in [14,15] are generalized.

Let $\mathcal{X}$ be a subcategory of mod-R. A subcategory $\mathcal{H}$ of $\mathcal{X}$ is called a *generator* for $\mathcal{X}$ if, for each object $X \in \mathcal{X}$, there exists an exact sequence $0 \to X' \to H \to X \to 0$ such that H is an object in $\mathcal{H}$ and $X' \in \mathcal{X}$. Dually, a *cogenerator* for a subcategory can be defined. The subcategory $\mathcal{H}$ is an *Ext-injective generator* for $\mathcal{X}$ if $\mathcal{H}$ is a generator for $\mathcal{X}$ and $Ext_R^{\geq 1}(X,H) = 0$ for each $X \in \mathcal{X}$ and each $H \in \mathcal{H}$. A module B is called a *generator* for $\mathcal{X}$ if $\mathrm{add}B$ is a generator for $\mathcal{X}$. Section 3 is devoted to discuss the relation between $\mathcal{X}(B)$-transposes and B-transposes when B is an Ext-injective generator for $\mathcal{X}$, where the B-transpose is the abbreviation of the $\mathrm{add}B(B)$-transpose. Let R be a left and right noetherian ring and $\mathcal{X}$ be a subcategory of mod-R closed under extensions