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On semigroup rings with decreasing Hilbert function

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Abstract

Given a one-dimensional semigroup ring $R = k[[S]]$, in this article we study the behaviour of the Hilbert function H_R. By means of the notion of *support* of the elements in S, for some classes of semigroup rings we give conditions on the generators of S in order to have decreasing H_R. When the embedding dimension v and the multiplicity e verify $v+3 \leq e \leq v+4$, the decrease of H_R gives an explicit description of the Apéry set of S. In particular for $e = v+3$, we prove that H_R is non-decreasing if $e \leq 12$ and we classify the semigroups with $e = 13$ and H_R decreasing. Finally we deduce that H_R is non-decreasing for every Gorenstein semigroup ring with $e \leq v+4$. This fact is not true in general: through numerical duplication and some of the above results another recent paper shows the existence of infinitely many one-dimensional Gorenstein rings with decreasing Hilbert function.

Keywords : Numerical semigroup, Monomial curve, Hilbert function, Apéry set.
Mathematics Subject Classification : Primary : 13H10 ; Secondary: 14H20 .

0 Introduction.

Given a local noetherian ring $(R, \mathfrak{m}, k)$ and the associated graded ring $G = \oplus_{n \geq 0}(\mathfrak{m}^n/\mathfrak{m}^{n+1})$, a classical hard topic in commutative algebra is the study of the Hilbert function H_R, defined as $H_R(n) = dim_k(\mathfrak{m}^n/\mathfrak{m}^{n+1})$, $n \in \mathbb{N}$. If $depth(G)$ is large enough, the values of this function can be determined by the Hilbert function of a lower dimensional ring, furthermore when G is Cohen Macaulay H_R is a non decreasing function; but even if R is Cohen Macaulay, in general G does not have this property. In particular, for a Cohen Macaulay one-dimensional local ring R we can have $depth(G) = 0$ and in this case H_R can be decreasing, i.e. $H_R(n) < H_R(n-1)$ for some n, see for example [11], [14], [15]. This fact cannot happen if the multiplicity e and the embedding dimension v of R satisfy either $v \leq 3$, or $v \leq e \leq v+2$, see [10], [8], [9], [20]. When $e \geq v+3$, several examples show that the function H_R can be decreasing.

In this article we deal with semigroup rings and in this case the "minimal decreasing" example we know, written in [18] and here recalled in (1.6), has $e = 13 = v+3$. Under the assumption $R = k[[S]]$, when G is not Cohen Macaulay, the study of certain subsets of S, called D_k and C_k, $k \in \mathbb{N}$, supplies a useful method to evaluate H_R; it has been applied in some recent papers [16], [4], [6]. We know several classes of semigroup rings with non-decreasing Hilbert function; this fact is true in particular when

(1) S is generated by an almost arithmetic sequence; if the sequence is arithmetic, then G is Cohen Macaulay [21],[17]

(2) S is four-generated and either belongs to some classes of symmetric semigroups [1], or has Buchsbaum tangent cone [4]

(3) S is balanced [16], [4]

(4) S is obtained by techniques of gluing numerical semigroups [2], [12]

(5) S satisfies certain conditions on the subsets D_k and C_k [6, Theorem 2.3, Corollary 2.4, Corollary 2.11].

The aim of this paper is the study of semigroup rings $R = k[[S]]$ having decreasing Hilbert function. To this goal we introduce and use the notion of *support* of the elements in S (1.3.4); by means of this tool in the Appendix we develop a technical analysis of the subsets D_k and C_k. In Section 2, through this machinery, under suitable assumptions on the Apéry set of S, we find conditions on S in order to have decreasing Hilbert function, see Theorem 2.1, Proposition 2.4 and Theorem 2.6. These results allow to construct classes of semigroup rings with decreasing H_R as shown in Examples 2.2 and 2.7 (where $e-7 \leq v \leq e-3$); especially, the Hilbert function in Example (2.7.1) decreases at two different levels.

In Section 3 we apply the above theorems to the semigroups with $v \in \{e-3, e-4\}$. For $v = e-3$ we show that the decrease of H_R is characterised by a particular structure of the sets D_2, C_2, C_3 and that H_R does not decrease for $e \leq 12$, see Theorem 3.2 and Proposition 3.3. In addition, for $e = 13$ we identify precisely the semigroups with H_R decreasing, see Proposition 3.6 and Example 3.7. In case $v = e-4$, we obtain analogous informations on the structure of C_2, C_3, D_2, D_3, see Theorems 3.9 and 3.10.

Another consequence of some of the above facts is that the semigroups with $|C_2| = 3$, $|C_3 \cap \textit{Apéry set}| \leq 1$ and H_R decreasing cannot be symmetric: in particular, in Corollary 3.11 we prove that every Gorenstein semigroup