Journal of Algebra 483 (2017) 1–36

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On maximal subalgebras

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ARTICLE INFO

Article history:
Received 4 September 2015
Available online 7 April 2017
Communicated by Luchezar L. Avramov

MSC:
primary 13B02, 13B30, 13G05, 13A18
secondary 14R10, 14H50

Keywords:
Commutative algebra
Integral domains
Valuations

ABSTRACT

Let $\mathbf{k}$ be an algebraically closed field. We classify all maximal $\mathbf{k}$-subalgebras of $\mathbf{k}[t, t^{-1}, y]$. To the authors' knowledge, this is the first such classification result for a commutative algebra of dimension > 1. Moreover, we classify all maximal $\mathbf{k}$-subalgebras of $\mathbf{k}[t, y]$ that contain a coordinate of $\mathbf{k}[t, y]$. Furthermore, we give plenty examples of maximal $\mathbf{k}$-subalgebras of $\mathbf{k}[t, y]$ that do not contain a coordinate.

1. Introduction

All rings in this article are commutative an have a unity. A *minimal ring extension* is a non-trivial ring extension that does not allow a proper intermediate ring. A good overview of minimal ring extensions can be found in [10]. A first general treatment of minimal ring extensions was done by Ferrand and Olivier in [4]. They came up with the following important property of minimal ring extensions.

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http://dx.doi.org/10.1016/j.jalgebra.2017.03.021
0021-8693/

Theorem 1.0.1 *(see [4, Théorème 2.2]). Let $A \subsetneq R$ be a minimal ring extension and let $\varphi\colon \operatorname{Spec}(R) \to \operatorname{Spec}(A)$ be the induced morphism on spectra. Then there exists a unique maximal ideal $\mathfrak{m}$ of A such that φ induces an isomorphism*

$$\operatorname{Spec}(R) \setminus \varphi^{-1}(\mathfrak{m}) \xrightarrow{\simeq} \operatorname{Spec}(A) \setminus \{\mathfrak{m}\}\,.$$

Moreover, the following statements are equivalent:

i) The morphism $\varphi\colon \operatorname{Spec}(R) \to \operatorname{Spec}(A)$ is surjective;
ii) The ring R is a finite A-module;
iii) We have $\mathfrak{m} = \mathfrak{m}R$.

Let $A \subsetneq R$ be a minimal ring extension. Then A is called a *maximal subring* of R. In the case where $\operatorname{Spec}(R) \to \operatorname{Spec}(A)$ is non-surjective, we call A an *extending*[1] maximal subring of R and otherwise, we call it a *non-extending*[2] maximal subring. Moreover, the unique maximal ideal $\mathfrak{m}$ of A (from the theorem above) is called the *crucial maximal ideal*.

In the non-extending case, Dobbs, Mullins, Picavet and Picavet-L'Hermitte gave in [3] a classification of all finite minimal ring extensions $A \subsetneq R$ based on the classification of all minimal ring extensions $A \subsetneq R$ where A is a field, due to Ferrand and Olivier [4]. Therefore, to some extent, the non-extending case is solved.

Our guiding problem is the following.

Problem. Classify all maximal subalgebras of a given affine $\mathbf{k}$-domain where $\mathbf{k}$ is an algebraically closed field.

Let $\mathbf{k}$ be an algebraically closed field and let R be an affine $\mathbf{k}$-domain. If R is one-dimensional and if $\operatorname{Spec}(R)$ contains more than one "smooth point at infinity", then the extending maximal subalgebras of R correspond bijectively to the "smooth points at infinity" of $\operatorname{Spec}(R)$. In fact, to every such point p at infinity, the subalgebra of functions in R that are defined in p is an extending maximal subalgebra of R and every extending maximal subalgebra of R is of this form. This is proven in Section 3 by using the Krull–Akizuki-Theorem.

In dimension two, the most natural algebra to study is the polynomial algebra in two variables $\mathbf{k}[t, y]$. Using the classification of extending maximal subalgebras of a one-dimensional affine $\mathbf{k}$-domain, we give in Section 4 plenty examples of extending maximal subalgebras of $\mathbf{k}[t, y]$ that do not contain a coordinate of $\mathbf{k}[t, y]$, i.e. they do not contain a polynomial in $\mathbf{k}[t, y]$ which is the component of an automorphism of $\mathbb{A}^2_{\mathbf{k}}$. These examples indicate that it is difficult to classify *all* extending maximal subalgebras of $\mathbf{k}[t, y]$.

[1] Since [4] proves that in this case f is a flat epimorphism, the literature calls this sometimes the "flat epimorphism case".
[2] The literature calls this sometimes the "finite case".