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Ideal containments under flat extensions

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ABSTRACT

Let $\varphi : S = k[y_0, ..., y_n] \to R = k[y_0, ..., y_n]$ be given by $y_i \to f_i$ where $f_0, ..., f_n$ is an R-regular sequence of homogeneous elements of the same degree. A recent paper shows for ideals, $I_\Delta \subseteq S$, of matroids, Δ, that $I_\Delta^{(m)} \subseteq I^r$ if and only if $\varphi_*(I_\Delta)^{(m)} \subseteq \varphi_*(I_\Delta)^r$ where $\varphi_*(I_\Delta)$ is the ideal generated in R by $\varphi(I_\Delta)$. We prove this result for saturated homogeneous ideals I of configurations of points in $\mathbb{P}^n$ and use it to obtain many new counterexamples to $I^{(rn-n+1)} \subseteq I^r$ from previously known counterexamples.

1. Introduction

Let R be a commutative Noetherian domain. Let I be an ideal in R. We define the mth symbolic power of I to be the ideal

$$I^{(m)} = R \cap \bigcap_{P \in Ass_R(I)} I^m R_P \subseteq R_{(0)}.$$

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In this note we shall be interested in symbolic powers of homogeneous ideals of 0-dimensional subschemes in $\mathbb{P}^n$. In the case that the subscheme is reduced, the definition of the symbolic power takes a rather simple form by a theorem of Zariski and Nagata [11] and does not require passing to the localizations at various associated primes. Let $I \subseteq k[\mathbb{P}^n]$ be a homogeneous ideal of reduced points, $p_1, ..., p_l$, in $\mathbb{P}^n$ with k a field of any characteristic. Then $I = I(p_1) \cap \cdots \cap I(p_l)$ where $I(p_i) \subseteq k[\mathbb{P}^n]$ is the ideal generated by all forms vanishing at p_i, and the mth symbolic power of I is simply $I^{(m)} = I(p_1)^m \cap \cdots \cap I(p_l)^m$.

In [10], Ein, Lazarsfeld and Smith proved that if $I \subseteq k[\mathbb{P}^n]$ is the radical ideal of a 0-dimensional subscheme of $\mathbb{P}^n$, where k is an algebraically closed field of characteristic 0, then $I^{(mr)} \subseteq (I^{(r+1-n)})^m$ for all $m \in \mathbb{N}$ and $r \geq n$. Letting $r = n$, we get that $I^{(mn)} \subseteq I^m$ for all $m \in \mathbb{N}$. Hochster and Huneke in [15] extended this result to all ideals $I \subseteq k[\mathbb{P}^n]$ over any field k of arbitrary characteristic.

In [5] Bocci and Harbourne introduced a quantity $\rho(I)$, called the resurgence, associated to a nontrivial homogeneous ideal I in $k[\mathbb{P}^n]$, defined to be $\sup\{s/t : I^{(s)} \nsubseteq I^t\}$. It is seen immediately that if $\rho(I)$ exists, then for $s > \rho(I)t$, $I^{(s)} \subseteq I^t$. The results of [10,15] guarantee that $\rho(I)$ exists since $I^{(mn)} \subseteq I^m$ implies that $\rho(I) \leq n$ for an ideal I in $k[\mathbb{P}^n]$. For an ideal I of points in $\mathbb{P}^2$, $I^{(mn)} \subseteq I^m$ gives $I^{(4)} \subseteq I^2$. According to [5] Huneke asked if $I^{(3)} \subseteq I^2$ for a homogeneous ideal I of points in $\mathbb{P}^2$. More generally Harbourne conjectured in [3] that if $I \subseteq k[\mathbb{P}^n]$ is a homogeneous ideal, then $I^{(rn-(n-1))} \subseteq I^r$ for all r. This led to the conjectures by Harbourne and Huneke in [13] for ideals I of points that $I^{(mn-n+1)} \subseteq \mathfrak{m}^{(m-1)(n-1)} I^m$ and $I^{(mn)} \subseteq \mathfrak{m}^{m(n-1)} I^m$ for $m \in \mathbb{N}$.

The second conjecture remains open. Cooper, Embree, Ha and Hoeful give a counterexample in [7] to the first for $n = 2 = m$ for a homogeneous ideal $I \subseteq k[\mathbb{P}^2]$. The ideal I in this case is $I = (xy^2, yz^2, zx^2, xyz) = (x^2, y) \cap (y^2, z) \cap (z^2, x)$ whose zero locus in $\mathbb{P}^2$ is the 3 coordinate vertices of $\mathbb{P}^2$, $[0 : 0 : 1]$, $[0 : 1 : 0]$ and $[1 : 0 : 0]$ together with 3 infinitely near points, one at each of the vertices, for a total of 6 points. Clearly the monomial $x^2y^2z^2 \in (x^2, y)^3 \cap (y^2, z)^3 \cap (z^2, x)^3$ so $x^2y^2z^2$ is in $I^{(3)}$. Note $xyz \in I$ so $x^2y^2z^2 \in I^2$, but $x^2y^2z^2 \notin \mathfrak{m}I^2$.

Shortly thereafter a counterexample to the containment $I^{(3)} \nsubseteq I^2$ was given by Dumnicki, Szemberg and Tutaj-Gasinska in [9]. In this case I is the ideal of the 12 points dual to the 12 lines of the Hesse configuration. The Hesse configuration consists of the 9 flex points of a smooth cubic and the 12 lines through pairs of flexes. Thus I defines 12 points lying on 9 lines. Each of the lines goes through 4 of the points, and each point has 3 of the lines going through it. Specifically I is the saturated radical homogeneous ideal $I = (x(y^3 - z^3), y(x^3 - z^3), z(x^3 - y^3)) \subset \mathbb{C}[\mathbb{P}^2]$. Its zero locus is the 3 coordinate vertices of $\mathbb{P}^2$ together with the 9 intersection points of any 2 of the forms $x^3 - y^3$, $x^3 - z^3$ and $y^3 - z^3$. The form $F = (x^3 - y^3)(x^3 - z^3)(y^3 - z^3)$ defining the 9 lines belongs to $I^{(3)}$ since for each point in the configuration, 3 of the lines in the zero locus of F pass through the point, but $F \notin I^2$ and hence $I^{(3)} \nsubseteq I^2$. (Of course this also means that $I^{(3)} \nsubseteq \mathfrak{m}I^2$.) More generally, $I = (x(y^n - z^n), y(x^n - z^n), z(x^n - y^n))$ defines a configuration of $n^2 + 3$ points called a Fermat configuration [1]. For $n \geq 3$, we again have $I^{(3)} \nsubseteq I^2$ [14,17] over any field of characteristic not 2 or 3 containing n distinct nth roots of 1.