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Definable envelopes in groups having a simple theory ☆

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ABSTRACT

Let G be a group having a simple theory. For any nilpotent subgroup N of class n, there is a definable nilpotent subgroup E of G which is virtually 'nilpotent of class at most $2n$' and finitely many translates of which cover N. The group E is definable using parameters in N, and normalised by $N_G(N)$. If S is a soluble subgroup of G of derived length ℓ, there is a definable soluble subgroup F which is virtually 'soluble of derived length at most 2ℓ' and contains S. The group F is definable using parameters in S and normalised by $N_G(S)$. Analogous results are shown in the more general setting where the ambient group G is defined by the conjunction of infinitely many formulas in a structure having a simple theory. In that case, the envelopes E and F are defined by the conjunction of infinitely many formulas.

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1. Introduction

When studying a group, a model theorist focuses on sets that are definable by formulas. It happens that in a group G, one finds a subgroup H of particular interest, having a given property P such as abelian, nilpotent, soluble etc. One then tries to find a definable group which also has property P and contains H. We call any group containing H an *envelope* of H. Finding a definable envelope of H with property P is possible when the ambient group is well behaved:

A group has the property MC if it satisfies the minimality condition on centralisers, that is if every strictly decreasing chain of centralisers $C_G(A_1) \supset C_G(A_2) \supset \cdots$ has a finite length. An abelian subgroup of an MC group is contained in an abelian definable subgroup (the centre of its centraliser). Stable groups are particular examples of MC groups. Poizat showed that if G is stable, then every nilpotent subgroup of G is contained in a nilpotent definable subgroup of the same nilpotency class, and every soluble subgroup of G is contained in a soluble definable subgroup of the same derived length (see [22]; the results also appear in [28]). In a recent paper, Altınel and Baginski have shown that a nilpotent subgroup of an MC group is enveloped by a nilpotent definable subgroup of the same nilpotency class (see [2]).

Wider than the class of stable groups is the class of groups that do not have the independence property. In the case where G is a group without the independence property, Shelah [26] has shown that if there is an infinite abelian subgroup of G, then there is one which is definable. This was improved by Aldama [1] who showed that any nilpotent subgroup of G of nilpotency class n is enveloped by a definable nilpotent group of class n. In those two cases, the parameters needed to define the enveloping group may lie in a saturated extension of the ambient group, *i.e.* the envelope is of the form $G \cap \mathbf{E}$, where $\mathbf{E}$ is a definable subgroup of a saturated extension $\mathbf{G}$ of G.

Another important class of groups extending the class of stable groups is the class of groups having a simple theory, which includes in particular all pseudofinite simple groups (*i.e.* pseudofinite groups without non-trivial normal subgroups, see [31] and [10]; we shall keep those two wordings to avoid any confusion between simple groups and groups having a simple theory). The previous results however do not hold in general if G has merely a simple theory. For instance they do not hold if G is an infinite extra-special p-group for some odd prime p, *i.e.* if every g in G has order p and in addition the centre of G is cyclic of order p and equals G'. Such a group has a simple theory (actually its theory is supersimple of SU-rank 1, see [11] and Appendix), is nilpotent of class 2, and possesses an infinite abelian subgroup; but any abelian definable subgroup of G is finite by [20].

However, if G is a group having a supersimple theory of finite rank which eliminates the quantifier $\exists^\infty$, it is shown in [6] that a soluble subgroup S of G of derived length ℓ is contained in a definable soluble subgroup E such that the derived series of S and E share the same number of infinite factors. The authors of [6] derive that the soluble radical of G is definable and soluble. If G merely has a simple theory, it is shown in