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Homotopy and commutativity principle

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ABSTRACT

We give evidence for the principle that a (special linear, symplectic, orthogonal) matrix over a commutative ring which is homotopic to the identity will commute up to an elementary matrix with all elements of the (respective) group.

1. Introduction

R will denote a commutative ring with $1 \neq 0$ in this article, unless stated otherwise.

The subject of injective stability for the linear group (i.e. $K_1(R)$) began in the famous paper of Bass–Milnor–Serre ([8]) where it was shown, in essence, that large sized stably elementary matrices were actually elementary matrices. This was shown by showing that the sequence (of pointed sets)

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$$\cdots \longrightarrow \frac{SL_n(R)}{E_n(R)} \longrightarrow \frac{SL_{n+1}(R)}{E_{n+1}(R)} \longrightarrow \cdots$$

stabilizes. The estimate they got was $n = 3$, when $\dim(R) = 1$, and for $n \geq \max\{3, d+3\}$ otherwise. They conjectured that the correct bound for the linear quotients should be $n \geq \max\{3, d+2\}$; which was established by L.N. Vaserstein in [39].

In [36] A.A. Suslin established the normality of the elementary linear subgroup $E_n(R)$ in $GL_n(R)$, for $n \geq 3$. This was a major surprise at that time as it was known due to the work of P.M. Cohn in [12] that in general $E_2(R)$ is not normal in $GL_2(R)$. This is the initial precursor to study the non-stable K_1 groups $\frac{SL_n(R)}{E_n(R)}$, $n \geq 3$.

This theorem can also be got as a consequence of the Local–Global Principle of D. Quillen (for projective modules) in [26]; and its analogue for the linear group of elementary matrices $E_n(R[X])$, when $n \geq 3$ due to A. Suslin in [36]. In fact, in [9] it is shown that, in some sense, the normality property of the elementary group $E_n(R)$ in $SL_n(R)$ is equivalent to having a Local–Global Principle for $E_n(R[X])$.

In [6], A. Bak proved the following beautiful result:

Theorem 1.1. *(A. Bak) For an almost commutative ring R with identity with center $C(R)$. The group $\frac{SL_n(R)}{E_n(R)}$ is nilpotent of class atmost $\delta(C(R))+3-n$, where $\delta(C(R)) < \infty$ and $n \geq 3$, where $\delta(C(R))$ is the Bass–Serre-dimension of $C(R)$.*

This theorem, which is proved by a localization and completion technique, which evolved from an adaptation of the proof of the Suslin's K_1-analogue of Quillen's Local–Global Principle, was the starting point of our investigation. In this paper, we show (see Corollary 2.20)

Theorem 1.2. *Let R be a local ring, and let $A = R[X]$. Then the group $\frac{SL_n(A)}{E_n(A)}$ is an abelian group for $n \geq 3$.*

This theorem is a simple consequence of the following principle (see Theorem 2.19):

Theorem 1.3. *(Homotopy and commutativity principle): Let R be a commutative ring. Let $\alpha \in SL_n(R)$, $n \geq 3$, be homotopic to the identity. Then, for any $\beta \in SL_n(R)$, $\alpha\beta = \beta\alpha\varepsilon$, for some $\varepsilon \in E_n(R)$.*

This principle is a consequence of the Quillen–Suslin Local–Global principle; and using a *non-symmetric* application of it as done by A. Bak in [6].

The existence of a Local–Global Principle enables us to prove similar results in various groups.

We restrict ourselves to the classical symplectic, orthogonal groups (and their relative versions); and to the automorphism groups of a projective module (with a unimodular element), a symplectic module (with a hyperbolic summand), and an orthogonal module (with a hyperbolic summand).