Accepted Manuscript

Flat complexes, pure periodicity and pure acyclic complexes

Daniel Simson

PII: S0021-8693(17)30112-6
DOI: http://dx.doi.org/10.1016/j.jalgebra.2017.02.013
Reference: YJABR 16124

To appear in: *Journal of Algebra*

Received date: 19 September 2016

Please cite this article in press as: D. Simson, Flat complexes, pure periodicity and pure acyclic complexes, *J. Algebra* (2017), http://dx.doi.org/10.1016/j.jalgebra.2017.02.013

Flat complexes, pure periodicity and pure acyclic complexes ☆

Daniel Simson[1],

Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, ul. Chopina 12/18, 87-100 Toruń, Poland

Dedicated to Professor Christian U. Jensen on the occasion of his 80th birthday

Abstract

The paper can be viewed as an addition and extension of the recent paper of Emmanouil [J. Algebra 465(2016), 190-213]. Among others, an alternative functor category approach to Emmanouil′s results is presented. By applying an idea of Neeman [Invent. Math. 174(2008), 255-308] and the main results obtained there, we prove that a chain complex F in a locally finitely presented Grothendieck category $\mathcal{A}$ is pure acyclic if and only if any chain map $f : P \to F$ from a complex P of pure-projective objects in $\mathcal{A}$ to F is null-homotopic. As a consequence we prove that any pure periodic object in $\mathcal{A}$ is pure-projective. Moreover, we show that $\mathcal{A}$ is pure semisimple if and only if $\mathcal{A}$ has the pure QF-property, that is, every pure-injective object in $\mathcal{A}$ is pure-projective.

Keywords: Grothendieck category, pure exact sequence, chain complex, null-homotopic complex, pure-projective object, pure-injective object, pure periodic object
2010 MSC: 05C22, 05C50, 06A11, 15A63, 68R05, 68W30

1. Introduction

Following recent advances on chain complexes of flat modules (see Neeman [16], Christensen-Holm [3], and Emmanouil [5]) and old results by Kiełpiński [13], Stenström [29] and Mitchell [17], we study homotopy properties of pure acyclic chain complexes, pure homological dimensions, and pure-periodic complexes in the framework of relative homological algebra (see [6, 10]). One of the inspirations for our study are the problems and results obtained by Benson-Goodearl [1], Bouchiba-Khaloui [2], Christensen-Holm [3], Simson [20, 21], [24]-[27], and Stenström [29].

Here by purity we mean the purity in the sense of Cohn [4]. The reader is referred to the monographs by Enochs-Jenda [6] and Jensen-Lenzing [11], and to the nice survey article by Huisgen-Zimmermann [10], for an elementary and comprehensive introduction to the relative homological algebra, pure homological dimensions, algebraically compactness, and the model theoretical algebra technique related with the subject of the paper.

Throughout we denote by $\mathbb{Z}$ the set of integers, by R an associative ring with an identity element, and by $\mathcal{A}$ a locally finitely presented Grothendieck category, see Popescu [18]. We recall from [9, 18] and [19]-[29] the basic facts we use in this paper. An object L of $\mathcal{A}$ is said to be **finitely presented** if the additive functor $\mathrm{Hom}_{\mathcal{A}}(L, -) : \mathcal{A} \longrightarrow Ab$ commutes with filtered colimits.

A chain complex

$$(1.1) \qquad F: \quad \ldots \to F_{n-1} \xrightarrow{\partial^F_{n-1}} F_n \xrightarrow{\partial^F_n} F_{n+1} \to \xrightarrow{\partial^F_{n+1}} F_{n+2} \to \ldots$$

in $\mathcal{A}$ is said to be **pure acyclic** if F is acyclic (i.e., $\mathrm{Ker}\,\partial^F_n = \mathrm{Im}\,\partial^F_{n-1}$, for all $n \in \mathbb{Z}$) and the induced chain complex

$$\ldots \to \mathrm{Hom}_{\mathcal{A}}(P, F_{n-1}) \to \mathrm{Hom}_{\mathcal{A}}(P, F_n) \to \mathrm{Hom}_{\mathcal{A}}(P, F_{n+1}) \to \ldots$$

of abelian groups is acyclic, for any finitely presented object P.

[1]Corresponding author.
Email addresses: simson@mat.umk.pl