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Exponential Sums and Riesz Energies

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Abstract

We bound an exponential sum that appears in the study of irregularities of distribution (the low-frequency Fourier energy of the sum of several Dirac measures) by geometric quantities: a special case is that for all $\{x_1, \dots, x_N\} \subset \mathbb{T}^2$, $X \geq 1$ and a universal $c > 0$

$$\sum_{i,j=1}^{N} \frac{X^2}{1 + X^4 \|x_i - x_j\|^4} \lesssim \sum_{\substack{k \in \mathbb{Z}^2 \\ \|k\| \leq X}} \left| \sum_{n=1}^{N} e^{2\pi i \langle k, x_n \rangle} \right|^2 \lesssim \sum_{i,j=1}^{N} X^2 e^{-cX^2 \|x_i - x_j\|^2}.$$

Since this exponential sum is intimately tied to rather subtle distribution properties of the points, we obtain nonlocal structural statements for near-minimizers of the Riesz-type energy. For $X \gtrsim N^{1/2}$ both upper and lower bound match for maximally-separated point sets satisfying $\|x_i - x_j\| \gtrsim N^{-1/2}$.

Keywords: Riesz energy, Discrepancy, Exponential sums, Fejér kernel
2010 MSC: 11L07, 42B05, 52C35 (primary), 74G65 (secondary)

1. Introduction and Main Results

1.1. Introduction

Let $\{x_1, \dots, x_N\} \subset \mathbb{T}^2$ (throughout this paper normalized to $\mathbb{T}^2 \cong [0,1]^2$). Montgomery's theorem [27] (see also Beck [4, 5]) is a classical example of an irregularity of distribution phenomenon: there exists a disk $D \subset \mathbb{T}^2$ with radius 1/4 or 1/2 such that the number of elements in the disk substantially deviates from its expectation

$$\left| \# \left\{ 1 \leq i \leq N : x_i \in D \right\} - N|D| \right| \gtrsim N^{1/4}.$$