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Sufficient conditions for large Galois scaffolds

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ABSTRACT

Let L/K be a finite, Galois, totally ramified p-extension of complete local fields with perfect residue fields of characteristic $p > 0$. In this paper, we give conditions, valid for any Galois p-group $G = \mathrm{Gal}(L/K)$ (abelian or not) and for K of either possible characteristic (0 or p), that are sufficient for the existence of a Galois scaffold. The existence of a Galois scaffold makes it possible to address questions of integral Galois module structure, which is done in a separate paper [BCE]. But since our conditions can be difficult to check, we specialize to elementary abelian extensions and extend the main result of [Eld09] from characteristic p to characteristic 0. This result is then applied, using a result of Bondarko, to the construction of new Hopf orders over the valuation ring $\mathfrak{O}_K$ that lie in $K[G]$ for G an elementary abelian p-group.

1. Introduction

Let p be prime, κ be a perfect field of characteristic p, and K be a local field with residue field κ. Let L be a totally ramified Galois extension of K with $G = \mathrm{Gal}(L/K)$

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of degree p^n for some $n > 0$, and let $\mathfrak{O}_L$ be the ring of integers of L (*i.e.* its valuation ring). *Local integral Galois module theory* asks a question that is a consequence of three classical results: the Normal Basis Theorem, which states that L *is free* over the group algebra $K[G]$; a result of E. Noether [Noe32], which concludes that, because L/K is wildly ramified, $\mathfrak{O}_L$ *is not free* over the group ring $\mathfrak{O}_K[G]$; and a local version of a result of H.W. Leopoldt [Leo59], which states that for absolute abelian extensions of the p-adic numbers (*i.e.* $K = \mathbb{Q}_p$), $\mathfrak{O}_L$ *is free* over its associated order

$$\mathfrak{A}_{L/K} = \{\alpha \in K[G] : \alpha\mathfrak{O}_L \subseteq \mathfrak{O}_L\},$$

the largest $\mathfrak{O}_K$-order in the group algebra $K[G]$ for which $\mathfrak{O}_L$ is a module.

Question 1.1. When is the ring of integers $\mathfrak{O}_L$ free over its associated order $\mathfrak{A}_{L/K}$?

Restrict for the moment to the situation where K is a finite extension of $\mathbb{Q}_p$. The earliest answers here showed us that unless $K = \mathbb{Q}_p$, $\mathfrak{O}_L$ need not be free over $\mathfrak{A}_{L/K}$, which is why the question is currently asked in this way. Additionally, those early answers suggested a form that we might expect the answers to take. Based upon work of F. Bertrandias and M.-J. Ferton [BF72] when L/K is a C_p-extension, and B. Martel [Mar74] when L/K is a $C_2 \times C_2$-extension, we might expect the answer to Question 1.1, necessary and sufficient conditions for $\mathfrak{O}_L$ to be free over $\mathfrak{A}_{L/K}$, to be expressed in terms of the ramification numbers associated with the extension (integers i such that $G_i \neq G_{i+1}$ where G_i is the ith ramification group [Ser79, IV §1]). There have not been that many further results in this direction. Still,

(1) When L/K is an abelian extension, and the ring of integers is replaced with the inverse different $\mathfrak{D}^{-1}_{L/K}$, [Byo97, Theorem 3.10] determines necessary conditions, in terms of ramification numbers, for the inverse different to be free over its associated order.
(2) When $K/\mathbb{Q}_p$ is unramified and L/K is a totally ramified abelian extension (not necessarily of p-power degree), D. Burns [Bur91] investigated freeness of ideals in $\mathfrak{O}_L$ over their associated orders in $K[G]$. This was extended in [Bur00] to the case where $K/\mathbb{Q}_p$ can be ramified, but associated orders are considered in $\mathbb{Q}_p[G]$ (or, more generally, in $E[G]$, where $E \subseteq K$ and $E/\mathbb{Q}_p$ is unramified). In both these situations, the existence of *any* ideal free over its associated order forces strong restrictions on the ramification of the extension L/K.
(3) When L/K is a special type of cyclic Kummer extension, namely $L = K(\sqrt[p^n]{1+\beta})$ for some $\beta \in K$ with $p \nmid v_K(\beta) > 0$, where v_K is the normalized valuation on K, Y. Miyata determines necessary and sufficient conditions for $\mathfrak{O}_L$ to be free over $\mathfrak{A}_{L/K}$ in terms of $v_K(\beta)$. These conditions can be restated in terms of ramification numbers [Miy98].
(4) Finally, we move into characteristic p with $K = \kappa((t))$. When L/K is a special type of elementary abelian extension, namely *near one-dimensional*, and thus has a *Galois*