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On holomorphic projection for symplectic groups ☆

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We construct certain Casimir operators and study the spectral properties of their resolvents on $L^2(\Gamma\backslash \mathrm{Sp}_2(\mathbb{R}))$. We define non-holomorphic multi-variable Poincaré series of exponential type for symplectic groups and continue them analytically in case of genus two for the small weight four using the above resolvents. We apply our results to describe the holomorphic projection to the weight four holomorphic discrete series in terms of Fourier coefficients by using Sturm's operator. This paper is a study of a prototype for symplectic groups and special orthogonal groups.

Contents

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1. Introduction

Poincaré series, whose origins date back to [22], were first systematically studied from a metrical point of view by Petersson [21]. For elliptic modular forms he considered holomorphic Poincaré series P_τ (of a certain weight κ) indexed by integers τ, for which the Petersson scalar product $\langle P_\tau, f\rangle$ with an arbitrary holomorphic cusp form f of weight κ is the τ's Fourier coefficient a_τ of f. Hence, as a consequence, the Poincaré series P_τ generate the space of cusp forms of weight κ. This approach for the construction of modular forms was generalized to the case of symplectic groups $G = \mathrm{Sp}_r(\mathbb{R})$ by Klingen [7] and Panchishkin [20]. To guarantee the convergence of the Poincaré series, there exists a natural lower bound on the weight κ. For instance in case of the symplectic groups $\mathrm{Sp}_r(\mathbb{R})$, the weight has to be larger than $2r$. In the further development, Sturm [29,30] went on to evaluate arbitrary L^2-functions against modular forms f, and thereby obtained an explicit integration operator for the holomorphic projection of f in terms of Fourier coefficients. Notice that the more recently defined Poincare series of [3,31,28,18], defined by cross summation of matrix coefficients also in the spirit of Poincaré, partially serve different purposes and do have different behavior.

Besides the mentioned reasons, the study of Poincaré series is often motivated by arithmetic applications, for example as tools for the p-adic interpolation of L-series for large weight [20]; or in Gross–Zagier's work [2] on the derivative of L-series at the critical point. In the latter case, for L-functions of elliptic curves the relevant weight is two, which is below the convergence bound of Poincaré series. In [2] this is overcome by 'Hecke's trick', i.e. one introduces an additional complex variable s to assure the convergence on some half plane $\mathrm{Re}\, s >> 0$ and then uses analytic continuation to the critical point $s = 0$. For the latter some explicit knowledge of the structure of Fourier coefficients as function of s is exploited in [2].

Arithmetic questions similar to those of Gross–Zagier also occur for higher rank groups, where e.g. the analog of the elliptic weight 2 case corresponds to the case of weight $r + 1$ for the groups $\mathrm{Sp}_r(\mathbb{R})$. For higher rank cases the approach of [2] based on the computation of Fourier coefficients no longer seems to be possible. Therefore our approach tries to extend the well-known strategy for Eisenstein series introduced by Roelcke [23–26] and Selberg [27]. Here the analytic continuation is achieved by Hilbert space techniques within the space $L^2(\Gamma\backslash G)$. In contrast to Eisenstein series studied by Roelcke and Selberg, our Poincare series are not eigenfunctions of the selfadjoint Casimir operators (generators of the center of the universal enveloping Lie algebra G) for all s. Although at first this looks like a disadvantage, we exactly use this fact as a substi-