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Pair correlations and equidistribution ☆

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A deterministic sequence of real numbers in the unit interval is called *equidistributed* if its empirical distribution converges to the uniform distribution. Furthermore, the limit distribution of the pair correlation statistics of a sequence is called *Poissonian* if the number of pairs $x_k, x_l \in (x_n)_{1 \le n \le N}$ which are within distance s/N of each other is asymptotically $\sim 2sN$. A randomly generated sequence has both of these properties, almost surely. There seems to be a vague sense that having Poissonian pair correlations is a "finer" property than being equidistributed. In this note we prove that this really is the case, in a precise mathematical sense: a sequence whose asymptotic distribution of pair correlations is Poissonian must necessarily be equidistributed. Furthermore, for sequences which are not equidistributed we prove that the square-integral of the asymptotic density of the sequence gives a lower bound for the asymptotic distribution of the pair correlations.

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1. Introduction

Let $(x_n)_{n\geq 1}$ be a sequence of real numbers. We say that this sequence is *equidistributed* or *uniformly distributed modulo one* if asymptotically the relative number of fractional parts of elements of the sequence falling into a certain subinterval is proportional to the length of this subinterval. More precisely, we require that

$$\lim_{N\to\infty} \frac{1}{N} \#\Big\{1 \leq n \leq N : \ \{x_n\} \in [a,b]\Big\} = b - a$$

for all $0 \leq a \leq b \leq 1$, where $\{\cdot\}$ denotes the fractional part. This notion was introduced in the early twentieth century, and received widespread attention after the publication of Hermann Weyl's seminal paper *Über die Gleichverteilung von Zahlen mod. Eins* in 1916 [16]. Among the most prominent results in the field are the facts that the sequences $(n\alpha)_{n\geq 1}$ and $(n^2\alpha)_{n\geq 1}$ are equidistributed whenever $\alpha \notin \mathbb{Q}$, and the fact that for any distinct integers $n_1, n_2, \ldots$ the sequence $(n_k\alpha)_{k\geq 1}$ is equidistributed for almost all α. All of these results were already known to Weyl, and can be established relatively easily using the famous *Weyl criterion*, which links equidistribution theory with the theory of exponential sums. For more background on uniform distribution theory, see the monographs [4,8]. We note that when $(X_n)_{n\geq 1}$ is a sequence of independent, identically distributed (i.i.d.) random variables having uniform distribution on $[0, 1]$, then by the strong law of large numbers this sequence is almost surely equidistributed. Consequently, in a very vague sense equidistribution can be seen as an indication of "pseudorandom" behavior of a deterministic sequence.

The investigation of pair correlations can also be traced back to the beginning of the twentieth century, when such quantities appeared in the context of statistical mechanics. In our setting, when $(x_n)_{n\geq 1}$ are real numbers in the unit interval, we define a function $F_N : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0} \cup \{\infty\}$ by

$$F_N(s) = \frac{1}{N} \# \left\{1 \leq m, n \leq N,\ m \neq n : \ \|x_m - x_n\| \leq \frac{s}{N}\right\}, \qquad (1)$$

and set

$$F(s) = \lim_{N\to\infty} F_N(s),$$

provided that such a limit exists; here $s \geq 0$ is a real number, and $\|\cdot\|$ denotes the distance to the nearest integer. The function F_N counts the number of pairs (x_m, x_n), $1 \leq m, n \leq N,\ m \neq n$, of points which are within distance at most s/N of each other (in the sense of the distance on the torus). If $F(s) = 2s$ for all $s \geq 0$, then we say that the asymptotic distribution of the pair correlations of the sequence is *Poissonian*. Again, one can show that an i.i.d. random sequence (sampled from the uniform distribution on $[0, 1]$) has this property, almost surely. Questions concerning the distribution of pair correlations of sequences such as $(\{n\alpha\})_{n\geq 1}$ or $(\{n^2\alpha\})_{n\geq 1}$ are linked with statistical

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