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Non-vanishing of fundamental Fourier coefficients of paramodular forms

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We prove that paramodular newforms of odd square-free level have infinitely many non-zero fundamental Fourier coefficients.

1. Introduction

The purpose of this article is to shed some light on Fourier coefficients of cuspidal paramodular forms. Paramodular forms are Siegel modular forms of degree 2 that are invariant under the action of the paramodular group

$$\Gamma^{\mathrm{para}}(N) := \mathrm{Sp}_4(\mathbb{Q}) \cap \begin{pmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z}/N \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{pmatrix}$$

for some natural number N.

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One of the most natural questions one may ask about a Siegel modular form F of degree 2 is its determination by certain 'useful' subset of Fourier coefficients. We are interested in an infinite subset

$$\{a(F,T) : \operatorname{disc} T = \text{ fundamental discriminant}\}$$

of fundamental Fourier coefficients, which plays an important role in the theory of Bessel models and L-functions. For instance, in certain cases, non-vanishing of a fundamental Fourier coefficient of a cuspidal Siegel modular form F is equivalent to existence of a global Bessel model of fundamental type (cf. [16, Lemma 4.1]) and is used to show analytic properties and special value results for L-functions for $\mathrm{GSp}_4 \times \mathrm{GL}_2$ associated to various twists of F (e.g. [8,12,17,18]). It is also known [19] that fundamental Fourier coefficients determine cuspidal Siegel modular forms of degree 2 of full level. Our result extends previous work by Saha [16, Theorem 3.4], [19, Theorem 1] and Saha, Schmidt [20, Theorem 2] in case of the levels $\mathrm{Sp}_4(\mathbb{Z})$ and $\Gamma_0^{(2)}(N)$.

Theorem. *Let $F \in S_k(\Gamma^{para}(N))$ be a non-zero paramodular cusp form of an arbitrary integer weight k and odd square-free level N which is an eigenfunction of the operators $T(p) + T(p^2)$ for primes $p \nmid N$, $U(p)$ for $p \mid N$ and μ_N. Then F has infinitely many non-zero fundamental Fourier coefficients.*

In particular, our theorem holds for paramodular newforms in the sense of [15].

Paramodular forms were already an object of interest of Siegel [21] but have become a true centre of attention within last ten years when Brumer and Kramer [4] conjectured an extension of the modularity theorem to abelian surfaces, known now as the paramodular conjecture.

Paramodular Conjecture. *There is a one to one correspondence between isogeny classes of abelian surfaces $\mathcal{A}/\mathbb{Q}$ of conductor N with $\mathrm{End}_{\mathbb{Q}}\mathcal{A} = \mathbb{Z}$ and (up to scalar multiplication) weight 2 cuspidal paramodular newforms F that are not Gritsenko lifts and have rational Hecke eigenvalues. Furthermore, the Hasse–Weil L-function of $\mathcal{A}$ is equal to the spinor L-function of F.*

In subsequent years the paramodular conjecture has been supported by an extensive computational evidence (e.g. [3,4,13]). Moreover, it was proved in the case when $\mathcal{A}$ is the Weil restriction of an elliptic curve with respect to real quadratic extension of $\mathbb{Q}$ (thanks to [10] and [7]), and in [2] some progress was made towards Weil restrictions with respect to imaginary quadratic extensions of $\mathbb{Q}$.

The proof of the above theorem consists of two parts and follows the strategy used in [16,19,20]. First we show that F has a non-zero primitive Fourier coefficient. This allows us to construct a non-zero modular form of half-integral weight which satisfies the assumptions of Theorems 2.2, 2.3 and therefore has infinitely many non-zero Fourier