ARTICLE IN PRESS
JPAA:5664

Journal of Pure and Applied Algebra ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Local Euler obstructions of toric varieties

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ARTICLE INFO

Article history:
Received 15 November 2016
Received in revised form 8 March 2017
Available online xxxx
Communicated by G.G. Smith

MSC:
Primary: 14M25; secondary: 52B20; 14B05; 14C17

ABSTRACT

We use Matsui and Takeuchi's formula for toric A-discriminants to give algorithms for computing local Euler obstructions and dual degrees of toric surfaces and 3-folds. In particular, we consider weighted projective spaces. As an application we give counterexamples to a conjecture by Matsui and Takeuchi. As another application we recover the well-known fact that the only defective normal toric surfaces are cones.

1. Introduction

The local Euler obstruction was used by MacPherson [20] in his construction of Chern classes for singular varieties. For a variety X the local Euler obstruction is a constructible function $\mathrm{Eu} : X \to \mathbb{Z}$ which takes the value 1 at smooth points of X. It is related to the Chern–Mather class and to the Chern–Schwartz–MacPherson class of X (see Remark 2.1).

Several equivalent definitions of the local Euler obstruction have been given, such as Kashiwara's definition of the local characteristic for a germ of an irreducible analytic space [17]. The first algebraic formula was given by González-Sprinberg and Verdier [10]. Matsui and Takeuchi use a topological definition [21], which defines the local Euler obstruction of X inductively using the Whitney stratification of X. They use this definition to prove a formula for the local Euler obstruction on a (not necessarily normal) toric variety X. In this article we will apply this formula to compute the local Euler obstructions of toric varieties of dimension ≤ 3.

For a normal toric surface X, we have that X is smooth if and only if $\mathrm{Eu}(X) = \mathbb{1}_X$ [21, Cor. 5.7]. Matsui and Takeuchi [21] conjecture that the corresponding statement should also hold for a higher dimensional normal and projective toric variety. As an application we present counterexamples to this conjecture.

A motivation for studying local Euler obstructions comes from formulas for the degrees of dual varieties. Given a projective variety $X \subset \mathbb{P}^N$, its dual variety $X^\vee \subset \mathbb{P}^{N^\vee}$ is the closure of the set of hyperplanes

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http://dx.doi.org/10.1016/j.jpaa.2017.04.016

JPAA:5664

$H \in \mathbb{P}^{N^\vee}$ such that there exists a smooth point $x \in X$ with $T_xX \subset H$. Generally $X^\vee$ will be a hypersurface in $\mathbb{P}^{N^\vee}$. Finding its equation is usually very difficult, but there are results which give the degree. Gelfand, Kapranov and Zelevinsky [8] proved a combinatorial formula for the degree of the dual variety of an embedded smooth toric variety. Matsui and Takeuchi generalized this formula to singular toric varieties, by weighting the terms by the local Euler obstruction. We will use this to describe algorithms to compute the degree of the dual variety of some toric varieties, in particular weighted projective spaces of dimension ≤3.

There has been recent interest in the local Euler obstruction. Aluffi studied Chern–Mather and Chern–Schwartz–MacPherson classes in [1]. Helmer and Sturmfels studied polar degrees and the local Euler obstruction in [14] related to the problem of finding the Euclidean distance degree of a variety. This problem is closely related to the contents of the current paper, since the Euclidean distance degree is expressible in terms of polar degrees, which in turn is expressible in terms of Matsui and Takeuchi's formulas involving the local Euler obstruction. In particular Helmer and Sturmfels study codimension one toric varieties [14, Thm. 3.7], and also they briefly study surfaces.

In Section 2 we define the local Euler obstruction. We recall some basic facts about toric varieties.

In Section 3 we present Matsui and Takeuchi's method for computing the local Euler obstruction of toric varieties and the degree of dual varieties.

In Section 4 we introduce our main examples of study, the weighted projective spaces. We describe them via toric geometry.

In Section 5 we follow Chapter 5 of [22] and apply the theory to toric surfaces. This relates to Hirzebruch–Jung continued fractions and the minimal resolution of singularities. We then do explicit computations for weighted projective planes.

In Section 6 we consider the local Euler obstruction of toric 3-folds. We prove that for a toric 3-fold $X_{P\cap M}$ with isolated singularities, the local Euler obstruction is always greater than or equal to 1. We find counterexamples to a conjecture by Matsui and Takeuchi [21, p. 2063].

In Section 7 we apply the above to describe which toric surfaces are dual defective, and to say something about which 3-dimensional weighted projective spaces are dual defective.

In the appendix we collect some computations of the local Euler obstruction and degrees of dual varieties for some weighted projective spaces.

2. The local Euler obstruction

Given a complex projective variety X of dimension d, consider the (generalized) Grassmann variety $\mathrm{Grass}_d(\Omega^1_X)$ representing locally free rank d quotients of Ω^1_X. The Nash blowup $\tilde{X}$ of X is the closure of the image of the morphism $X_{\mathrm{sm}} \to \mathrm{Grass}_d(\Omega^1_X)$. Let $\pi\colon \tilde{X} \to X$ denote the projection. The Nash sheaf $\tilde{\Omega}$ is the restriction to $\tilde{X}$ of the tautological rank d sheaf on $\mathrm{Grass}_d(\Omega^1_X)$. There is a surjection $\pi^*\Omega^1_X \to \tilde{\Omega}$, and the Nash blowup is universal with respect to birational morphisms $f : Y \to X$ such that there is a locally free sheaf $\mathcal{F}$ of rank d on Y and a surjection $f^*\Omega^1_X \to \mathcal{F}$. Let $\tilde{T}$ denote the dual of $\tilde{\Omega}$.

The local Euler obstruction of a point $x \in X$ is the integer

$$\mathrm{Eu}(x) = \int\limits_{\pi^{-1}(x)} c(\tilde{T}|_{\pi^{-1}(x)}) \cap s(\pi^{-1}(x), \tilde{X}).$$

On the smooth locus of a variety the local Euler obstruction takes the value 1. It is a local invariant, thus we can compute it on an open affine cover.

This is the usual algebraic definition, used by amongst others [7, Ex. 4.2.9]. When the ambient variety is clear we will simply write Eu for the local Euler obstruction, however if there are different ambient varieties we sometimes write Eu_X for the local Euler obstruction on X.

Please cite this article in press as: B.I.U. Nødland, Local Euler obstructions of toric varieties, J. Pure Appl. Algebra (2017), http://dx.doi.org/10.1016/j.jpaa.2017.04.016