Journal of Pure and Applied Algebra 222 (2018) 636–673

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Rewriting in higher dimensional linear categories and application to the affine oriented Brauer category

Clément Alleaume

Univ Lyon, Université Claude Bernard Lyon 1, CNRS UMR 5208, Institut Camille Jordan, 43 blvd. du 11 novembre 1918, F-69622 Villeurbanne cedex, France

ARTICLE INFO

Article history:
Received 30 June 2016
Received in revised form 27 February 2017
Available online 7 June 2017
Communicated by J. Adámek

ABSTRACT

In this paper, we introduce a rewriting theory for linear monoidal categories. Those categories are a particular case of linear (n, p)-categories that we define in this paper. We also define linear (n, p)-polygraphs, a linear adaptation of n-polygraphs, to present linear $(n-1, p)$-categories. We focus then on linear $(3, 2)$-polygraphs to give presentations of linear monoidal categories. We finally give an application of this theory to prove a basis theorem on the category $\mathcal{AOB}$. Our method uses decreasingness, a property introduced by van Ostroom.

1. Introduction

Affine walled Brauer algebras were introduced by Rui and Su [10] in the study of super Schur–Weyl duality. They show the Schur–Weyl duality between general Lie superalgebras and affine walled Brauer algebras. A linear monoidal category, the affine oriented Brauer category $\mathcal{AOB}$ was introduced in [2] to encode each walled Brauer algebra as one of its morphism spaces. This category is used to prove basis theorems for the affine walled Brauer algebras given in [10]. More precisely, the authors provides an explicit basis for each affine walled Brauer algebra. The proof of this theorem uses an intermediate result on cyclotomic quotients of $\mathcal{AOB}$. For each of those quotients, a basis is given. With these multiple bases, each morphism space of $\mathcal{AOB}$ is given a generating family which is proved to be linearly independent.

Our aim is to give a constructive proof of the mentioned basis result. For this, we study $\mathcal{AOB}$ in this article by rewriting methods. Rewriting is a model of computation presenting relations between expressions as oriented computation steps. There are multiple examples of rewriting systems. An abstract rewriting system [5] is the data made of a set S and a relation $\rightarrow$ on S called the rewrite relation. A rewriting sequence from a to b is a finite sequence $(u_0, u_1, \cdots, u_{n-1}, u_n)$ of elements of S such that:

$$a = u_0,$$

E-mail address: clement.alleaume@univ-st-etienne.fr.

http://dx.doi.org/10.1016/j.jpaa.2017.05.002
0022-4049/

$$b = u_n,$$

and for any $0 \leqslant k < n$, the relation $u_k \to u_{k+1}$ holds. A word rewriting system is the data made of an alphabet A and a relation $\Rightarrow$ on the free monoid A^* over A. We say that there is a rewriting step from a word u to a word v if there are words a, b, u' and v' such that:

$$u = au'b,$$
$$v = av'b,$$
$$u' \Rightarrow v'.$$

A higher dimensional generalization of such rewriting systems has been introduced by Burroni [3] under the name of polygraph. An $(n+1)$-polygraph is a rewriting system on the n-cells of an n-category.

To study $\mathcal{AOB}$ from a rewriting point of view, we will need to introduce the rewriting systems presenting monoidal linear categories. The objects giving such rewriting systems will be called linear (n, p)-polygraphs which are linear adaptations of n-polygraphs. Linear monoidal categories are a special case of what we will call linear (n, p)-categories. In this language, linear monoidal categories are linear $(2, 2)$-categories with only one 0-cell. They are presented by linear $(3, 2)$-polygraphs. Once those objects are defined, we will introduce the rewriting theory of linear $(3, 2)$-polygraphs. Thanks to this theory, we will construct bases for the morphism spaces of $\mathcal{AOB}$.

Rewriting can offer constructive proofs by giving presentations of objects with certain properties. For example, two crucial properties studied in rewriting systems are termination and confluence. A rewriting system is terminating if it has no infinite rewriting sequence, in which case all computations end. A rewriting system is confluent if any pair of rewriting sequences with the same source can be completed into a pair of rewriting sequences with the same target, in which case all computations lead to the same result. A rewriting system is said to be convergent if it is terminating and confluent. In the case of word rewriting, the property of convergence gives a way to decide the word problem, that is, deciding if two words in the free monoid A^* over A are equal in the quotient of A^* by the relation $\Rightarrow$.

What we will do in the case of $\mathcal{AOB}$ is giving a confluent presentation $\overline{\mathrm{AOB}}$ of this linear $(2, 2)$-category with some others properties. Those properties will prove that the families proposed in [2] are indeed bases. The linear $(3, 2)$-polygraph $\overline{\mathrm{AOB}}$ will not be terminating, which will prevent us to prove that $\overline{\mathrm{AOB}}$ is confluent by using Newman's lemma, a criterion needing termination to prove confluence from a weaker property called local confluence [5]. To prove that $\overline{\mathrm{AOB}}$ is confluent, we will use a more general property called decreasingness introduced by van Ostroom in [13], see also [14]. We will prove that $\overline{\mathrm{AOB}}$ is decreasing and use the theorem from [13] stating decreasingness implies confluence.

In the first section, we start by recalling the notions of higher dimensional category theory. Then, we define linear (n, p)-categories, which will be our higher dimensional categories with linear structure. After defining them, we recall in the second section the categorical construction of the category of n-polygraphs given in [7]. We define next the categorical construction of the category of linear (n, p)-polygraphs. We give their main rewriting properties, such that 4.2.15 in the case $(n, p) = (3, 2)$ in which $\mathcal{AOB}$ falls.

In the third section we will study the decreasingness property defined in the case of abstract rewriting systems by van Ostroom [13]. Then, in the last section, we recall from [2] the definition of the linear $(2, 2)$-category $\mathcal{AOB}$. This will lead us to give two linear $(3, 2)$-polygraphs presenting $\mathcal{AOB}$. Those linear $(3, 2)$-polygraphs will be called AOB and $\overline{\mathrm{AOB}}$. The first one is a translation of the definition of $\mathcal{AOB}$.

The main result of this article, Theorem 5.2.9 states $\overline{\mathrm{AOB}}$ is confluent. It will be proved with the properties of confluence of critical branchings and decreasingness. This theorem gives us the main result of [2] as an entirely constructive consequence given as Corollary 5.2.10.