Journal of Pure and Applied Algebra 221 (2017) 948–959

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Quadratic descent of totally decomposable orthogonal involutions in characteristic two

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ARTICLE INFO

Article history:
Received 29 April 2016
Received in revised form 2 July 2016
Available online 20 August 2016
Communicated by V. Suresh

MSC:
16W10; 16K20; 11E39

ABSTRACT

We investigate the quadratic descent of totally decomposable algebras with involution of orthogonal type in characteristic two. Both separable and inseparable extensions are included.

1. Introduction

Let K/F be a field extension and let (A, σ) be a central simple algebra with involution over K. By a *descent* of (A, σ) to F we mean an F-algebra with involution (B, σ') satisfying $(A, \sigma) \simeq_K (B, \sigma') \otimes_F (K, \mathrm{id})$. In the case where $\operatorname{char} F \neq 2$ and K/F is a quadratic extension, the descent of orthogonal involutions of degree 2 and 4 over K was studied in [2] and necessary and sufficient conditions in terms of the discriminant and the Clifford algebra were given. Also, if $\operatorname{char} F = 2$ and K/F is a finite extension satisfying $K^2 \subseteq F$, a criterion for totally decomposable algebras with orthogonal involution over K to have a descent to F was obtained in [7, (6.2)].

Let (A, σ) be a totally decomposable algebra with orthogonal involution over a field F of characteristic 2. In [7], a maximal commutative Frobenius subalgebra $\Phi(A, \sigma)$ of A was introduced and the relation between this subalgebra and the *Pfister invariant* $\mathfrak{Pf}(A, \sigma)$, a bilinear Pfister form associated to (A, σ) defined in [3], was studied. It was also shown that totally decomposable orthogonal involutions can be classified, up to conjugation, by their Pfister invariant (see [7, (6.5)]).

In this work we study the quadratic descent of totally decomposable orthogonal involutions in characteristic 2. We first consider inseparable extensions. If $K = F(\sqrt{\alpha})$ is a quadratic extension of F and (A, σ) is a totally decomposable algebra with orthogonal involution over K, we find necessary and sufficient conditions

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http://dx.doi.org/10.1016/j.jpaa.2016.08.013
0022-4049/

for (A, σ), in terms of $\Phi(A, \sigma)$ and $\mathfrak{Pf}(A, \sigma)$, to have a descent to F. It is also shown that if (A, σ) has a descent to F, then it has a *totally decomposable descent* to F, i.e., a descent which is totally decomposable itself (see Lemma 3.4). This result complements [7, (6.2)] for quadratic extensions.

We next consider separable extensions. In §4 we investigate the quadratic descent of quaternion algebras with orthogonal involution. In §5 some properties of totally singular conic algebras are studied. These results will be used in §6 to obtain a descent property of the Pfister invariant. Finally, in §7 we state our main result Theorem 7.3 which asserts that for a separable quadratic extension K/F and a totally decomposable algebra with orthogonal involution (A, σ) over K, the following conditions are equivalent: (1) (A, σ) has a descent to F. (2) (A, σ) has a totally decomposable descent to F. (3) The corestriction $\mathrm{cor}_{K/F}(A)$ splits and $\mathfrak{Pf}(A, \sigma) \simeq \mathfrak{b}_K$ for some symmetric bilinear form $\mathfrak{b}$ over F. In the case where σ is isotropic, these conditions are also equivalent to: (4) $\mathrm{cor}_{K/F}(A)$ splits and $\Phi(A, \sigma) \simeq S \otimes_F K$ for some totally singular conic F-algebra S.

2. Preliminaries

Throughout this work, all fields are supposed to be of characteristic 2.

Let A be a central simple algebra over a field F. An *involution* on A is a map $\sigma : A \to A$ satisfying $\sigma^2(x) = x$, $\sigma(x + y) = \sigma(x) + \sigma(y)$ and $\sigma(xy) = \sigma(y)\sigma(x)$ for $x, y \in A$. For an algebra with involution (A, σ) we define the subspaces

$$\mathrm{Alt}(A, \sigma) = \{x - \sigma(x) \mid x \in A\} \quad \text{and} \quad \mathrm{Sym}(A, \sigma) = \{x \in A \mid \sigma(x) = x\}.$$

If $\sigma|_F = \mathrm{id}$, we say that σ is of *the first kind*. Otherwise, σ is said to be of *the second kind*. An involution σ of the first kind is called *symplectic* if after scalar extension to a splitting field, it becomes adjoint to an alternating bilinear form. Otherwise, it is called *orthogonal*. According to [5, (2.6 (2))], σ is orthogonal if and only if $1 \notin \mathrm{Alt}(A, \sigma)$. We denote the discriminant of an orthogonal involution σ by $\mathrm{disc}\,\sigma$ (see [5, (7.2)]). An algebra with involution (A, σ) (or the involution σ itself) is called *isotropic* if there exists a nonzero element $x \in A$ such that $\sigma(x)x = 0$. Otherwise, it is called *anisotropic*.

A *quaternion algebra* over a field F is a four-dimensional central simple F-algebra. Every quaternion algebra Q has a basis $(1, u, v, w)$, called a *quaternion basis*, satisfying $u^2 + u \in F$, $v^2 \in F^\times$ and $w = uv = vu + v$ (see [5, p. 25]). In this case, if $a = u^2 + u \in F$ and $b = v^2 \in F^\times$, then Q is denoted by $[a, b)_F$. If $b \in F^{\times 2}$ or $a \in \wp(F) := \{x^2 + x \mid x \in F\}$, then Q splits. Also, it is readily verified that every element $v' \in Q \setminus F$ satisfying $v'^2 \in F^\times$ extends to a quaternion basis $(1, u', v', w')$ of Q.

Let V be a finite-dimensional vector space over a field F. A symmetric bilinear form $\mathfrak{b} : V \times V \to F$ is called *metabolic* if there exists a subspace W of V with $\dim_F W = \frac{1}{2} \dim_F V$ such that $\mathfrak{b}|_{W \times W} = 0$. We say that two bilinear forms $\mathfrak{b}$ and $\mathfrak{b}'$ are *similar* if $\mathfrak{b} \simeq \lambda\mathfrak{b}'$ for some $\lambda \in F^\times$. For $\alpha \in F$ the isometry class of the symmetric bilinear form $\mathfrak{b}((x_1, x_2), (y_1, y_2)) = x_1y_1 + \alpha x_2y_2$ on F^2 is denoted by $\langle 1, \alpha\rangle$. Also, for $\alpha_1, \cdots, \alpha_n \in F^\times$, the form $\langle 1, \alpha_1\rangle \otimes \cdots \otimes \langle 1, \alpha_n\rangle$ is called a *bilinear (n-fold) Pfister form* and is denoted by $\langle\langle \alpha_1, \cdots, \alpha_n\rangle\rangle$. Finally, if K/F is a field extension and $\mathfrak{b}$ is a bilinear form over F, the scalar extension of $\mathfrak{b}$ to K is denoted by $\mathfrak{b}_K$.

3. Inseparable descent

We begin our discussion with a definition from [7].

Definition 3.1. An algebra A over a field F is called a *totally singular conic algebra* if $x^2 \in F$ for every $x \in R$.